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HADAMARD'S FUNDAMENTAL SOLUTION AND CONICAL REFRACTION*

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Abstract Conical refraction in anisotropic media shows two different light speeds, hence the characteristic conoid is composed of two sheets. In a special case that two of the dielectric constants are equal, conic refraction is depicted by a partial differential operator which is factorizable. Thus the singular support of the fundamental solution should also be composed of two sheets. In this paper, the author gives the Hadamard construction of the fundamental solution which is just singular on these two sheets. In case of conic refraction considered, these two sheets are tangent to each other along two bi-characteristic curves, and a special singularity of the boundary-layer type appears there.

Key Words Conic refraction; Hadamard fundamental solution; geometric-optical asymptotics; boundary-layer type singularity.
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1. Introduction

Conical refraction was first reported by E. Bartholinus in 1669 to the effect that a ray of light impinging normally on a face of Iceland spar is splitted into two rays thus violating Snell's law of refraction. C.Huygens failed to explain it but used it as a powerful support of his wave theory of light, see [1]. Later on, many physicists and mathematicians, most notably, W.R. Hamilton, devoted themselves to its investigation. For a comprehensive historical treatment, see Gårding [2]. Now we shall start from Maxwell's equations for electomagnetic field in anisotropic media, which states that the electric vector $\boldsymbol{u} = (u_1, u_2, u_3)$ satisfies the following system of PDE

$$\frac{\partial^2}{\partial t^2}(\Xi \boldsymbol{u}) = \Delta \boldsymbol{u} - \text{grad div}\boldsymbol{u},$$

$$\Xi = \begin{pmatrix} \mu\varepsilon_1/c^2 & 0 & 0 \\ 0 & \mu\varepsilon_2/c^2 & 0 \\ 0 & 0 & \mu\varepsilon_3/c^2 \end{pmatrix} \tag{1}$$

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where μ is the constant of permeability, $\varepsilon_i, i = 1,2,3$, the dielectric constants in the principal axial directions, i.e. principal axes of polarizing tensor, c, the light velocity. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) = (\mu\varepsilon_1/c^2, \mu\varepsilon_2/c^2, \mu\varepsilon_3/c^2)$.The characteristic form of (1) is

$$\begin{aligned} D(\xi,\tau) &= \begin{vmatrix} \sigma_1\tau^2 - \xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \sigma_2\tau^2 - \xi_3^2 - \xi_1^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \sigma_3\tau^2 - \xi_1^2 - \xi_2^2 \end{vmatrix} \\ &= \sigma_1\sigma_2\sigma_3\tau^2[\tau^4 - \Psi(\xi)\tau^2 - \rho^2\Phi(\xi)] \end{aligned} \tag{2}$$

where

$$\begin{aligned} \rho^2 &= \xi_1^2 + \xi_2^2 + \xi_3^2 \\ \Psi(\xi) &= \sum_{i=1}^{2} \sigma_i^{-1}(\rho^2 - \xi_i^2) \\ \Phi(\xi) &= (\sigma_1\sigma_2\sigma_3)^{-1}\sum_{i=1}^{3} \sigma_i\xi_i^2 \end{aligned} \tag{3}$$

Set $v = \dfrac{\partial^2 u_i}{\partial t^2}, i = 1,2,3$, it is easily seen that

$$\frac{\partial^4 v}{\partial t^4} - \Psi\left(\frac{\partial}{\partial x}\right)\frac{\partial^2 v}{\partial t^2} + \triangle \cdot \Phi\left(\frac{\partial}{\partial x}\right) v = 0 \tag{4}$$

In the special case when $\sigma_1 = \sigma_2 > \sigma_3$,

$$\begin{aligned} \Psi(\xi) &= (\sigma_1^{-1})(2\xi_3^2 + \xi_1^2 + \xi_2^2) + \sigma_3^{-1}(\xi_1^2 + \xi_2^2) \\ \Phi(\xi) &= (\sigma_1\sigma_3)^{-1}(\xi_1^2 + \xi_2^2) + \sigma_1^{-2}\xi\sigma_3^2 \end{aligned} \tag{5}$$

the characteristic form of (4) can be factored as follows

$$\tau^4 - \Psi(\xi)\tau^2 + \rho^2\Phi(\xi) = \left\{\tau^2 - \frac{1}{2}[\Psi(\xi) + X(\xi)]\right\}\left\{\tau^2 - \frac{1}{2}[\Psi(\xi) - X(\xi)]\right\}$$

with

$$X(\xi) = (\xi_1^2 + \xi_2^2)(\sigma_1 - \sigma_3)^{-1}$$

Thus, the equation (4) can be written as

$$\left[\frac{\partial^2}{\partial t^2} - \sigma_1^{-1}\triangle\right]\left[\frac{\partial^2}{\partial t^2} - \sigma_3^{-1}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \sigma_1^{-1}\frac{\partial^2}{\partial x_3^2}\right] v = 0 \tag{6}$$

It is conceivable that its characteristic conoid (with vertex at the origin) is composed of two sheets:

$$\begin{aligned} k_1(x,t) &\equiv t^2 - \sigma_1\left(x_1^2 + x_2^2 + x_3^2\right) = 0 \\ k_2(x,t) &\equiv t^2 - \sigma_3\left(x_1^2 + x_2^2\right) - \sigma_1 x_3^2 = 0 \end{aligned} \tag{7a}$$