J.Partial Diff. Eqs. 12(1999), 193-200
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# GLOBAL SOLUTIONS TO SOME QUASILINEAR PARABOLIC SYSTEMS IN POPULATION DYNAMICS

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(Received Mar. 8, 1997; revised Apr. 9, 1998)

**Abstract** In this present paper, using the duality technique and the Hölder's inequality, we study the global existence of the solutions to some quasilinear parabolic systems with the cross-diffusion effects in population dynamics.
**Key Words** Quasilinear parabolic systems; population dynamics; cross-diffusion; global solutions.
**Classification** 35K57.

## 1. Introduction

In this paper we study the existence and uniqueness of global solutions to the following parabolic systems in population dynamics

$$\begin{array}{ll} u_t = \operatorname{div}\,[\nabla(a_1u+duv)+b_1(\nabla\Phi(x))u]+uF(u,v), & \text{in } \Omega\times(0,\infty) \\ v_t = \operatorname{div}\,[\nabla(a_2v)+b_2(\nabla\Phi(x))v]+vG(u,v), & \text{in } \Omega\times(0,\infty) \\ \alpha_1u+(1-\alpha_1)\partial u/\partial n=0, & \text{on } \partial\Omega\times(0,\infty) \\ \alpha_2v+(1-\alpha_2)\partial v/\partial n=0, & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & \text{in } \Omega \end{array} \tag{1.1}$$

where, $\Omega$ is a bounded domain in $R^n$ $(n\geq 1)$ with smooth boundary $\partial\Omega$; $a_i>0$ $(i=1,2)$ are positive constants; $d$ and $b_i$ $(i=1,2)$ are nonnegative constants; $\Phi\in C^2(\overline{\Omega})$ is a given real function on $\overline{\Omega}$; $\partial/\partial n$ denotes the outer normal derivative on the boundary $\partial\Omega$. $u_0(x)$ and $v_0(x)$ are initial functions which are assumed to satisfy

$$\begin{array}{ll} u_0,v_0\in H^{1+\varepsilon}(\Omega) \text{ with some } \varepsilon>0, & \text{and } u_0,v_0\geq 0 \text{ in } \Omega \\ \alpha_1u_0+(1-\alpha_1)\partial u_0/\partial n=0, & \text{on } \partial\Omega\times(0,\infty) \\ \alpha_2v_0+(1-\alpha_2)\partial v_0/\partial n=0, & \text{on } \partial\Omega\times(0,\infty) \end{array} \tag{$H_0$}$$

also, $F,G\in C^1(R^2_+)$ are given functions on $R^2_+$; and

$$\alpha_1=\alpha_2=1;\ \text{or } \alpha_1=\alpha_2=0,\ \text{or } \alpha_1,\alpha_2\in(0,1) \tag{1.2}$$

This system (1.1) has been introduced by Shigesada et al. [1] as a model of the population dynamics in $\Omega$ of two competitive species, $u$ and $v$ are population densities of two species and $F$ (resp. $G$) represents the growth rate of $u$-species (resp. $v$-species). As to the existence of global solutions to (1.1), some particular, but interesting, cases have been already studied by several authors (cf. [1]-[7]). In [5], Yamada followed the idea of Pozio and Tesei, discussed (1.1) in the framwork of $C(\Omega)$ and $L_p(\Omega)$, employed the operator theory of evolution equations in $L_p(\Omega)$. He took $F$ and $G$ of the forms

$$\begin{aligned} F(u,v) &= a - f(u) - \phi(v) \\ G(u,v) &= b - \psi(u) - g(v) \end{aligned} \tag{$*$}$$

with positive constant $a, b$ and made the following assumptions on $f, g, \phi, \psi$ and $\alpha_i$ $(i = 1, 2)$

① $f, g, \phi$ and $\psi$ are increasing functions of class $C^1[0,\infty)$ with $f(0) = g(0) = \phi(0) = \psi(0) = 0$;

② $g(v^*) = b$ with some $v^* > 0$;

③ $u \max_{0 \le \omega \le u} \psi(\omega) \le c_0 \psi(u)$ for all $u \ge 0$ with some $c_0 > 0$; $(**)$

④ $\phi(0) \to +\infty$ and $f(u)/\psi(u)^r \to +\infty$ as $u \to +\infty$ with some $r > 1$;

⑤ $\alpha_1 = \alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$.

His main result is the existence of bounded global solutions to (1.1). Also, in [6], Redlinger has considered the following case

(1) $F = e_1 - h(u) - a_1 v$, $G = e_2 - a_2 u - b_2 v$,

where $a_i, e_i$ and $b_2$ are positive constants, and the function $h$ is assumed to have the following properties:

a. $h \in C^2(R, R)$ and $\lim\limits_{s \to +\infty} \inf h(s)/s^r > 0$ for some $r > 0$ in the case $N = 1$ and $r > 1$ in the case $N > 1$,

b. $h \in O(s^k)$ as $s \to +\infty$ for some $k > 0$.

(2) $a_1 = a_2 = \delta \in C(\Omega, \{0, 1\})$.

Redlinger has shown the existence of the global attractor (and thus, in particular, global existence and boundedness of solutions).

## 2. Notations and Results

For convenience, we will use the standard notation $Q(T)$ for the $\Omega \times [0, T]$. We have the following standard local existence result.

**Proposition 2.1** In the function space $C([0,\infty); L_2(\Omega))$, there exists a unique nonnegative solution $\{u, v\}$ of (1.1) on $\Omega \times [0, T_0)$ for some $T_0 \in (0, \infty]$, such that, for any $0 < T < T_0$,

$$u, v \in C((0,T]; H^2(\Omega)) \cap C^1((0,T]; L_2(\Omega)).$$

For the proof of this result, we refer to [3, 7, 8]. We make the following assumptions on $F$, $G$ and $\Phi$:

① There exist $C_0 > 0, A > 0, B > 0$ are constants, and, $C_1(s), C_2(s) \in C(R_+^1)$, s.t.,

$$G(u,v) \le C_0$$