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Some Results on the Stability of Non-classical Shock Waves

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract Part 1 of this paper establishes the infinite-time stability of a class of over-compressive viscous shock waves; the equations studied here are a mathematical analogue of those of magnetohydrodynamics. Part 2 communicates a rather general short-time stability result for undercompressive shock waves in several space dimensions; technically, this is an easy extension of Majda's corresponding result for Laxian shock waves.

Key Words Conservation laws, shock waves, stability, overcompressive, under-compressive.

Classification 35L, 35K, 76W.

1. Infinite-time Stability of Non-classical Viscous Shock Waves

A traveling viscous shock wave solution

$$u^*(x,t) = \phi^*(x - st), \quad \phi^*(\pm\infty) = u^\pm \tag{1}$$

of a "viscous" system of n conservation laws

$$u_t + (f(u))_x = (B(u)u_x)_x \tag{2}$$

is called stable for infinite time if with some appropriate norm $\|\cdot\|$ and some $\delta > 0$, the following holds for any perturbation $\overline{u}_0 : \mathbf{R} \to \mathbf{R}^n$: If $\|\overline{u}_0\| < \delta$, then the solution u of (2) with data

$$u(x,0) = \phi^*(x) + \overline{u}_0(x), \quad x \in \mathbf{R} \tag{3}$$

exists for all times $t > 0$ and converges in the sense

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} |u(x,t) - \phi(x - st)| = 0 \tag{4}$$

to another viscous shock wave of profile ϕ with the same end states $\phi(\pm\infty) = u^\pm$. For classical shock waves, stability in this sense has been proved by Goodman, Matsumura and Nishihara, Liu, Szepessy and Xin in [1–4] under various assumptions; certain non-classical shock waves were shown to be stable by Liu and co-authors in [5–7]. The purpose of this paper consists in establishing an infinite-time stability result of certain non-classical viscous shock waves in the "cylindrical model" introduced by the author in [8]. This model (see also [9]) is given by the equations

$$\begin{aligned} y_t + (zy)_x &= \mu y_{xx} \\ z_t + \frac{1}{2}(|y|^2 + z^2)_x &= \zeta z_{xx} \end{aligned} \tag{5}$$

where $x \in \mathbf{R}$, $t \in [0,\infty)$, $y(x,t) \in \mathbf{R}^{n-1}$ $(n \geq 3)$, $z(x,t) \in \mathbf{R}$, and $\mu, \zeta > 0$. We abbreviate (5) as (2), with $u \equiv (y, z)$ and $B \equiv \mathrm{diag}\,(\mu, \cdots, \mu, \zeta) \in \mathbf{R}^{n\times n}$. The inviscid part of (5) is hyperbolic, with characteristic speeds

$$\lambda_{3/1}(u) = z \pm |y|, \quad \lambda_2(u) = z \tag{6}$$

We consider the maximally overcompressive case, i.e., shock waves which satisfy

$$\lambda_1(u^-) > s > \lambda_3(u^+) \tag{7}$$

A solution $\phi : \mathbf{R} \to \mathbf{R}^n$ of the boundary value problem

$$B\phi' = f \circ \phi - s\phi - q, \quad \phi(\pm\infty) = u^\pm \tag{8}$$

with

$$q = f(u^-) - su^- = f(u^+) - su^+ \tag{9}$$

will be called a profile for the pair (u^-, u^+). We write $q = (q_1, q_2)$, $q_1 \in \mathbf{R}^{n-1}$, and, w. l. o. g., fix from now on

$$s = 0 \quad \text{and} \quad q_2 > 0 \tag{10}$$

Lemma 1 *If $q_1 \in \mathbf{R}^{n-1}$ is sufficiently small, then there exists* (i) *a unique pair (u^-, u^+) with* (7), (9), *and* (ii) *a unique profile ϕ_0^* for (u^-, u^+) with and $\phi_0^*(0) = 0$.*

Proof (i) The equation $f(u) - su = q$ reads

$$\begin{aligned} zy &= q_1 \\ \frac{1}{2}(|y|^2 + z^2) &= q_2 \end{aligned} \tag{11}$$

If $q_1 = 0$, then its solution set consists of the two points

$$u^\pm = (0, \mp(2q_2)^{1/2})$$