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CONVERGENCE OF APPROXIMATE SOLUTIONS FOR QUASILINEAR HYPERBOLIC CONSERVATION LAWS WITH RELAXATION*

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Abstract In this article the author considers the limiting behavior of quasilinear hyperbolic conservation laws with relaxation, particularly the zero relaxation limit. Our analysis includes the construction of suitably entropy flux pairs to deduce the L^∞ estimate of the solutions, and the theory of compensated compactness is then applied to study the convergence of the approximate solutions to its Cauchy problem.

Key Words Compensated compactness; entropy flux pair; conservation laws; relaxation.

Classification 35L60, 35L65, 35B.

1. Introduction

The relaxation phenomenon arises in several areas of physics, such as dynamics, elastic dynamics, multiphase flow, phase transition, etc (See [1–3] for detail). It is important in many physical situations: in the kinetic theory the relaxation time is the mean free path and in viscoelasticity the relaxation time is the strength of memory, in gas dynamics the phenomenon occurs when the gas is in the thermo-nonequilibrium.

Consider the following quasilinear hyperbolic conservation laws with relaxation:

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x + \dfrac{v - V_*(u)}{\delta} = 0, \quad \delta > 0 \end{cases} \tag{1.1}$$

The first equation is a conservation law for u, while the second one contains a relaxation mechanism with $V_*(u)$ as the equilibrium value for v and δ is the relaxation time (See [4–6]).

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The rate term $\frac{V_*(u)-v}{\delta}$ tends to make v relax towards the equilibrium curve $V_*(u)$, acting as a source when v is less than $V_*(u)$ and as a sink otherwise. Thus, $\frac{V_*(u)-v}{\delta}$ introduces relaxation effects into the system (1.1). These effects add some amount of diffusion (See [4–6]).

When the solution to the system (1.1) is close to equilibrium, that is, $v = V_*(u)$, we often ignore the rate equation and replace the conservation law by the equilibrium equation

$$u_t + f(u, V_*(u))_x = 0 \tag{1.2}$$

In this paper we consider the following Cauchy problem of quasilinear hyperbolic conservation laws with relaxation:

$$\begin{cases} u_t - f(v)_x = 0 \\ v_t = g(u)_x = \dfrac{V_*(u)-v}{\delta}, \quad \delta > 0 \end{cases} \tag{1.3}$$

$$t = 0 : u = u_0(x), v = v_0(x) \tag{1.4}$$

Suppose that $(u^\varepsilon_\delta(t,x), v^\varepsilon_\delta(t,x))$ is the solution sequence of the following Cauchy problem of parabolic systems:

$$\begin{cases} u_t - f(v)_x = \varepsilon u_{xx}, \quad \varepsilon > 0 \\ v_t - g(u)_x = \dfrac{V_*(u)-v}{\delta} + \varepsilon v_{xx} \\ t = 0 : u = u_0(x), v = v_0(x) \end{cases} \tag{1.5}$$

corresponding to the Cauchy problem (1.3)–(1.4). In this paper we will prove that there exists a subsequence of $(u^\varepsilon_\delta(t,x), v^\varepsilon_\delta(t,x))$ (still denoted), such that

$$(u^\varepsilon_\delta(t,x), v^\varepsilon_\delta(t,x)) \to (u_\delta(t,x), v_\delta(t,x)), \quad \text{a.e.} \tag{1.6}$$

as $\varepsilon \to 0^+$, and $(u_\delta(t,x), v_\delta(t,x))$ is the weak solution to the Cauchy problem (1.3)–(1.4). Moreover, there exists a subsequence of $(u_\delta(t,x), v_\delta(t,x))$ (still denoted), such that

$$(u_\delta(t,x), v_\delta(t,x)) \to (u(t,x), v(t,x)), \quad \text{a.e.} \tag{1.7}$$

as $\delta \to 0^+$, and $(u(t,x), v(t,x))$ satisfies

(1) $v(t,x) = V_*(u(t,x))$, a.e., as $t > 0$;

(2) $u(t,x)$ is a unique entropy solution for the following Cauchy problem

$$\begin{cases} u_t - f(V_*(u))_x = 0 \\ t = 0 : u = u_0(x) \end{cases} \tag{1.8}$$