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GLOBAL SMOOTH SOLUTIONS TO A SYSTEM OF DISSIPATIVE NONLINEAR EVOLUTION EQUATIONS*

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Abstract The existence and uniqueness are proved for global classical solutions of the following initial-boundary problem for the system of parabolic equations which is proposed by Hsieh as a substitute for the Rayleigh-Benard equation and can lead to Lorenz equations:

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx} \\ \theta_t = -(1-\beta)\theta + \nu\psi_x + (\psi\theta)_x + \beta\theta_{xx} \\ \psi(0,t) = \psi(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0 \\ \psi(x,0) = \psi_0(x), \quad \theta(x,0) = \theta_0(x) \end{cases}$$

Key Words System of parabolic equations; nonlinear; initial-boundary problem; global classical solution.

Classification 35K10.

1. Introduction

The study of chaos has been an active field since Lorenz derived his famous equation by mode truncation from Rayleigh-Benard equations. However, because of its high nonlinearity, one does not know how chaos arises from the Rayleigh-Benard equations. In order to overcome this difficulty and understand the chaos phenomina, Hsieh [1] constructed a model of partial differential equations as a substitute for the Rayleigh-Benard equations:

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx} \\ \theta_t = -(1-\beta)\theta + \nu\psi_x + 2\psi\theta_x + \beta\theta_{xx} \end{cases}$$

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here σ, ν, α, and β are positive constants satisfying $\alpha < \sigma, \beta < 1$. Like the Rayleigh-Benard equations, Hsieh's equations also yield the Lorenz equations. However, Hsieh's system is much simpler than the former for theoretical and numerical analysis.

The conversation form of Hsieh's equations reads as

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx} \\ \theta_t = -(1 - \beta)\theta + \nu\psi_x + (\psi\theta)_x + \beta\theta_{xx} \end{cases} \tag{1.1}$$

With a similar truncation used by Lorenz in [2], the system (1.1) also leads to the Lorenz equations. In fact, taking

$$\begin{cases} \psi = \sqrt{2}X(t)\sin x \\ \theta = \sqrt{2}Y(t)\cos x + 2Z(t)\cos 2x \end{cases}$$

for (1.1) and retaining only the coefficients of $\sin x, \cos x$ and $\cos 2x$, one can obtain Lorenz equations. For the details, see Tang [3].

If we ignore the diffusion terms on the right-hand side of (1.1), then the characteristic of the remained system takes the form

$$\lambda = \frac{-\psi \pm \sqrt{\psi^2 - 4\sigma(\nu + \theta)}}{2}$$

This means when $|\psi|$ is small enough, the remained system is basically elliptic, so one may expect that the inherent instability will cause growth (Hsieh [4] has found that the instabilities of quite a few systems in fluid mechanics are associated with ellipticity) and drive the system into hyperbolic regime. However, the damping would diminish the amplitude and draw the system back into elliptic regime. So there might be a "switching back and forth" mechanism, which could possibly lead to some complicated behavior, even chaos.

Recently, Tang [3] performed extensive numerical simulation on the system (1.1), and found that the solutions to (1.1) decrease to zero for big dissipation. Moreover, for smaller dissipation, he found a route to chaos, i.e., the system (1.1) admits only steady state solutions represented by a set of discrete peaks, and the number of peaks increases as the dissipation decreases.

Nevertheless, there are few strictly theoretical results about (1.1). In fact, the existence and uniqueness of solutions to (1.1) have not been solved, except the result obtained by Tang in [3], which reads that if (1.1) does not admit a global smooth solution $\{\psi(x,t), \theta(x,t)\}$, then θ must blow-up at first (see [3; Theorem 2.2.3]).

As the first step of the program to establish a theoretical analysis on the chaotic nature of the solution for (1.1) with small dissipation, the present paper is devoted to