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ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

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Abstract We study the existence of positive solutions for a class of semilinear elliptic systems in general domains via the blow up argument and degree theory. The main idea can be used to establish the existence of positive solutions for the Navier problems of polyharmonic semilinear equations in general domains.
Key Words Semilinear elliptic systems; positive solutions; blow up argument; degree theory; polyharmonic semilinear equations.
Classification 35J65, 35J50, 35J55.

1. Introduction

In this paper we study the existence of positive solutions of the following systems

$$\begin{aligned} -\Delta u &= f(x,u,v,w) \\ -\Delta v &= g(x,u,v,w) \quad \text{in } \Omega \\ -\Delta w &= h(x,u,v,w) \\ u = v &= w = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{0}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ with smooth boundary and $f, g, h : \Omega \times \mathbf{R}^3 \to [0, +\infty)$ are given functions which we specify later.

In recent years, the problems of type (0) with two semilinear equations

$$\begin{aligned} -\Delta u &= f(x,u,v) \\ -\Delta v &= g(x,u,v) \quad \text{in } \Omega \\ u = v &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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have been studied by several authors by using different approaches. See, for example, [1-8].

In the case when there exists a Hamiltonian function $H : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(x,t,s) = H_s(x,t,s), \quad g(x,t,s) = H_t(x,t,s)$$

the problem (1) has a variational structure and thus under some suitable hypotheses on H it can be studied by variational methods. For the results of this type the interested reader may refer to [2, 7] and the references therein.

The problem (1) was also studied in a quite different method in [8] and [6, 1]. In [8], *a-priori* bounds of positive solutions are proved in the case that Ω is a ball and for a large class of Hamiltonians of the form

$$H(u,v) = F(u) + G(v)$$

while in [6] it is assumed that Ω is a convex domain and $F'' \geq 0$, $G'' \geq 0$. In both papers the *a-priori* bounds are obtained by using the integral identities proved in [9], [10] and [11]. For the additional results on *a-priori* bounds for positive solutions of semilinear elliptic equations and systems, see [12], [13], [3] and the references therein.

In the present paper we shall study (0) in a general bounded domain Ω with smooth boundary contained in $\mathbf{R}^N$. To prove the existence of positive solutions of (0) we first need to study some non-existence results for positive solutions for (0) in $\mathbf{R}^N$, then establish *a-priori* bounds for positive solutions for this problem. To deal with the *a priori* bounds we extend to the systems the so called blow up method introduced by Gidas and Spruck in their fundamental paper [14]. In [14], they used this approach to obtain *a priori* bounds of positive solutions for the single semilinear equations. Finally we use degree theory to obtain the existence. The main result is

Theorem A *Consider the system*

$$\begin{aligned} -\Delta u &= a(x)|v|^{\delta-1}v \\ -\Delta v &= b(x)|w|^{\theta-1}w \quad \text{in } \Omega \\ -\Delta w &= c(x)|u|^{\mu-1}u \\ u = v &= w = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2}$$

where $a, b, c : \Omega \to (0, +\infty)$ *are continuous functions such that*

$$\hat{a} = \min_{x\in\overline{\Omega}} a(x) > 0, \quad \hat{b} = \min_{x\in\overline{\Omega}} b(x) > 0, \quad \hat{c} = \min_{x\in\overline{\Omega}} c(x) > 0$$