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LOCAL EXISTENCE OF BOUNDED SOLUTIONS TO THE DEGENERATE STEFAN PROBLEM WITH JOULE'S HEATING *

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Abstract This paper deals with the degenerate Stefan problem with Joule's heating, which describes the combined effects of heat and electrical current flows in a metal. The local existence of a bounded weak solution for the problem in proved. Also a degenerate thermistor problem with discontinuous conductivity is considered.
Key Words Degenerate; bounded solutions; compactness; Stefan problem.
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1. Introduction

A multidimensional, two-phase problem of Stefan type that describes the processes of electric heating in a conducting material is considered. When an electrical current flows across the conductor, Joule heating is generated by the resistance of the conductor to the electrical current, which brings about the increase of the temperature. Once the melting temperature is crossed, latent heat will be absorbed and the phase change phenomena occurs.

Let Ω be a smooth bounded domain in $\mathcal{R}^N$, $N \geq 1$. The electrical potential and the temperature distribution inside $\Omega_T \equiv \Omega \times [0, T]$ are denoted by $\varphi = \varphi(x, t)$ and $u = u(x, t)$, respectively. Let u_*, be the melting temperature, which is a positive constant, $h = h(x, t)$ be the enthalpy density. Then the mathematical model under consideration is the follows:

Find a triplet $\{h, u, \varphi\}$, such that

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)| \nabla \varphi|^2 \quad \text{in } \Omega_T \tag{1.1}$$

$$\nabla \cdot (\sigma(u) \nabla \varphi) = 0 \quad \text{in } \Omega_T \tag{1.2}$$

$$h \subset \alpha(u) \quad \text{in } \Omega_T \tag{1.3}$$

$$u = u_0(x) \quad \text{on } \Omega \times \{0\} \tag{1.4}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T] \tag{1.5}$$

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$$\varphi = \varphi_0 \quad \text{on } \partial\Omega \times [0,T] \tag{1.6}$$

where $\sigma(u)$ is the temperature depending on electrical conductivity, $\alpha(u)$ is the maximal monotone graph modelling the phase change process,

$$\alpha(s) = \begin{cases} s - \lambda & \text{if } s < u_* \\ |u_* - \lambda, u_* + \lambda| & \text{if } s = u_* \\ s + \lambda & \text{if } s > u_* \end{cases} \tag{1.7}$$

For simplicity we have assumed that the temperature is equal to zero on $\partial\Omega$, and will assume $\lambda = 1$.

When $\lambda = 0$ (i.e., $h \equiv u$, no phase change occurs), the problem (1.1)–(1.6) is called the thermistor problem and has been studied by several authors; see, *e.g.*, [1–6]. For the physical background and the known results for the problem (1.1)–(1.6) with $\lambda = 1$, we refer to [7] for more details and the references therein. In [8] we have proved the existence of the C^0-solution in two-space dimension. The results mentioned above are proved by assuming that the conductivity is continuous and uniformly positive. When the conductivity $\sigma(s)$ has limit zero as $|s| \to \infty$, Xu [9] introduced a notion of capacity solution to avoid the difficulty which is caused by the fact that the boundedness of solution u has not been proved. In this paper we shall estimate the uniform bound of approximated temperature u in local time by using heat potential analysis and comparison principle, and then obtain a sequence of approximate solution converging in L^2 space to the local weak solution of (1.1)–(1.6) by using a generalized compactness lemma, which is the improvement of the well-known Lions-Aubin compactness lemma and is crucial to the proof of existence results for the Stefan-like problem. As a corollary the global existence of bounded weak solution for the problem (1.1)–(1.6) with uniformly positive conductivity is obtained. We note that the boundedness of weak solution has not been discussed in [7], and it seems that the proof of (4.21) in [7] is false.

The plan of the paper is as follows. In Section 2, we state the definition of the weak solution and main results, and prove two auxiliary lemmas. The one is a generalized compactness lemma, and the other one is a maximum principle for parabolic equation with the source in Morrey space $L^\infty(0,T; L^{1,N-2+2\alpha}(\Omega))$ (See Section 2 below). In Section 3 we introduce a family of regularized problems, to whom the existence and uniform estimate are proved. Next in Section 4 we will conclude that there exists a sequence of approximating solutions converging to the weak solution of (1.1)–(1.6). Finally in Section 5 a time-dependent thermistor problem with degenerate and discontinuous conductivity is considered and the local existence of bounded weak solution for the problem is obtained

2. Formulation of the Problem and Auxiliary Lemmas

Let us assume

$$\sigma(s) \in \mathrm{Lip}\,(\mathcal{R}^1),\ 0 < \sigma(s) \le \sigma_0 < +\infty, \quad \forall s \in \mathcal{R}^1,\ \lim_{|s|\to\infty} \sigma(s) = 0 \tag{2.1}$$