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ON THE CAUCHY PROBLEM AND INITIAL TRACE FOR NONLINEAR FILTRATION TYPE WITH SINGULARITY

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Abstract In this paper, we consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \qquad \text{in } \boldsymbol{R}^N \times (0,T]$$
$$u(x,0) = u_0(x) \quad \text{in } \boldsymbol{R}^N$$

where $\varphi \in C[0,\infty) \cap C^1(0,\infty)$, $\varphi(0) = 0$ and $\left(1 - \frac{2}{N}\right)^+ < a \le \frac{\varphi'(s)s}{\varphi(s)} \le m$ for some $a \in \left(\left(1-\frac{2}{n}\right)^+, 1\right)$, $s > 0$. The initial value $u_0(x)$ satisfies $u_0(x) \ge 0$ and $u_0(x) \in L^1_{\text{loc}}(\boldsymbol{R}^N)$. We prove that, under some further conditions, there exists a weak solution u for the above problem, and moreover $u \in C^{\alpha,\frac{\alpha}{2}}_{x,t_{\text{loc}}}(\boldsymbol{R}^N \times (0,T])$ for some $\alpha > 0$.
Key Words Filtration type; Cauchy problem; initial trace; existence of solutions.

Classification 35K65, 35D05.

1. Introduction

In this paper, we deal with the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta\varphi(u(x,t)) \text{ in } \boldsymbol{R}^N \times (0,T] \tag{1.1}$$
$$u(x,0) = u_0(x) \quad \text{in } \boldsymbol{R}^N \tag{1.2}$$

where $\varphi \in C[0,\infty) \cap C^1(0,\infty)$ satisfies $\varphi(0) = 0$, $\left(1 - \frac{2}{N}\right)^+ < a \le \frac{\varphi'(s)s}{\varphi(s)} \le m$ for some $a \in \left(\left(1-\frac{2}{N}\right)^+, 1\right)$ and $u_0(x) \ge 0$.

Equation (1.1) has been suggested as a mathematical model for a lot of physical problems, we will not recall them here.

When $\varphi(u) = u^m$ and $m > 1$, it was proved in [1] that if the function $u_0(x)$ satisfies

$$\sup_{R\ge 1} R^{-\left(\frac{2}{m-1}+N\right)} \int_{|x|\le R} u_0(x)dx < \infty \tag{1.3}$$

then (1.1), (1.2) has a weak solution. The condition (1.3) is also a sufficient one for (1.1), (1.2) possessing a weak solution (see [2]).

When $\varphi(u) = u$, the necessary and sufficient condition for (1.1), (1.2) possessing a weak solution is that $u_0(x)$ satisfies

$$\int_{\boldsymbol{R}^N} e^{-c|x|} u_0(x)dx < \infty \tag{1.4}$$

As for general $\varphi(u)$ with $1 \le a \le \dfrac{\varphi'(s)s}{\varphi(s)} \le m$, the necessary condition for (1.1), (1.2) possessing a weak solution is that $u_0(x)$ satisfies

$$|||u_0|||_r = \sup_{\rho \ge r} [\Phi^{-1}(\rho^2)]^{-1} \fint_{B_\rho} u_0 dx < \infty \tag{1.5}$$

for some $r > 0$. Here $\Phi(s) = \dfrac{\varphi(s)}{s}$ and $\fint_{B_\rho} u_0 dx = \dfrac{1}{|B_\rho|}\int_{B_\rho} u_0 dx$ (see [3]).

When $\varphi(u) = u^m$ with $0 < m < 1$, it is proved in [4] that (1.1), (1.2) has a weak solution when $u_0(x)$ is a nonnegative and locally integrable function on $\boldsymbol{R}^N$. Moreover, if $m \in \left(\left(1 - \dfrac{2}{N}\right)^+, 1\right)$, then the weak solutions are $C_{\text{loc}}^{\alpha, \frac{\alpha}{2}}$. But for general $\varphi(u)$ satisfying $a \le \dfrac{\varphi'(s)s}{\varphi(s)} \le m$ with $a < 1$, there is no result on this problem. In this paper, we concern the case when $\left(1 - \dfrac{2}{N}\right)^+ < a < 1$ and obtain a result similar to those mentioned above. We will prove the following theorems.

Theorem 1 *Suppose* $\varphi \in C^1(0, \infty) \cap C[0, \infty)$, $\varphi(0) = 0$, $\varphi(s) > 0$, *and* $a \le \dfrac{\varphi'(s)s}{\varphi(s)} \le m$ *when* $s > 0$, *for some* $a, m > 0$ *with* $a > \dfrac{m + (1 - \frac{2}{N})^+}{2}$, $u_0(x)$ *is a nonnegative and locally integrable function. If* u *is a weak solution of* (1.1) (1.2), *then there exist* $C_i = C_i(N, a, m)$, $i = 0, 1$ *and* $\varepsilon \in \left(\dfrac{(1 - \frac{2}{N})^+ + m}{2}, a\right]$ *with* $\alpha = 1 - \dfrac{m - \varepsilon}{1 - r_0} \cdot \dfrac{N}{2} > 0$, *such that for* $\rho > 0$, $x_0 \in \boldsymbol{R}^N$, $t \in (0, T_{\rho, x_0}]$, *we have*

$$\|u(*, t)\|_{\infty, B_\rho(x_0)} \le C_1 \left(\frac{\rho^2}{t}\right)^{\frac{1}{1 - r_0}\frac{N}{2}} \left[\fint_{B_{2\rho}(x_0)} (u_0 \vee k) dx\right]^{\frac{1}{1 - r_0}} \tag{1.6}$$

where

$$T_{\rho, x_0} = \frac{C_0 \rho^2}{\Psi^{\frac{1}{\alpha}}\left[\fint_{B_{2\rho}(x_0)} (u_0 \vee k) dx\right]} \tag{1.7}$$

$\Psi(s) = \dfrac{\varphi(s)}{s^\varepsilon}$, $r_0 = \dfrac{N}{2}(1 - \varepsilon)$ *and* $u_0 \vee k = \max(u_0, k)$, *for some* $k \ge 1$.

Theorem 2 *Assume that* $\varphi \in C^2(0, \infty) \cap C[0, \infty)$ *satisfies* $\varphi(0) = 0$, $\varphi(s) > 0$ *as* $s > 0$, *and*

$$a \le \frac{\varphi'(s)s}{\varphi(s)} \le m, \quad s > 0 \tag{1.8}$$