J.Partial Diff. Eqs. 9(1996), 139–152
©International Academic Publishers Vol.9, No.2

WEAKLY NONLINEAR OSCILLATORY WAVES WITH MULTI-PHASES IN IDEAL INCOMPRESSIBLE FLUID*

Chen Ming and Qiu Qingjiu
(Mathematics Department, Nanjing University, Nanjing 210008, China)
(Received Nov. 22, 1994; revised June 18, 1995)

Abstract In this paper, we apply the technique of weakly nonlinear geometric optics to study weakly nonlinear oscillatory waves with multi-phases in d-dimensional ideal incompressible fluid for $d \geq 2$. Precisely, we give a rigorous asymptotic expansion for the solution of the oscillatory initial value problem to the ideal incompressible Euler equations. Generally, this problem is not well posed. However, the coherence assumption and small divisor property imposed on the phases functions lead to a compatibility condition for the solvability of profile equations from which we can determine every profile.

Key Words Weakly nonlinear geometric optics; ideal incompressible fluids; oscillatory waves with multi-phases.
Classification 35B20, 35B10, 35F25, 35Q20.

1. Introduction

The method of weakly nonlinear geometric optics is one kind of nonlinear perturbation techniques, it is widely used to analyze the weakly nonlinear waves and their interactions in hyperbolic equations (see [1–8] etc.). For an introduction to the weakly nonlinear geometrical optics, one can consult [9]. One of the important aspects of the method is to study the oscillatory initial value problem for nonlinear hyperbolic equations. The original study of this aspect dates back to the work of P. Lax in 1957 in [10], where his results are about linear hyperbolic systems. In 1969, Y. Choquet-Bruhat generalized the results of P. Lax to nonlinear case (see [11]). Both of them dealt with oscillatory solutions with only a single phase. For the oscillatory solution with multi-phases, J-L. Joly, G. Métivier and J. Rauch studied the case for one dimensional nonlinear hyperbolic systems (see [1]). For the oscillatory solutions with multi-phases of multidimensional nonlinear systems, the contrary example constructed in [3] shows that this problem is not well posed due to the appearance of many complicate phenomena such as focusing effects. Usually, some strong geometric hypotheses on the set of phases are needed to prevent these phenomena from appearing (see [2], [6]).

*The project supported by National Natural Science Foundation of China.

In this paper, we use the method of nonlinear geometric optics to study the weakly nonlinear oscillatory waves in arbitrary dimensional ideal incompressible fluid. More precisely, we give a rigorous asymptotic expansion of the solution for the initial problem with oscillatory initial data. We note that the system of equations for ideal incompressible fluid may be regarded as the degenerate case to the system of the ideal compressible fluid by setting the density to be constant and the type of the latter is strictly hyperbolic. For the system of ideal incompressible fluid, although it is studied in many papers (see [12–14] etc.), little is known about its oscillatory initial problem as far as the authors know.

We arrange the paper as follows:

In the second section we reduce the incompressible system to a system of singular integrodifferential equations in which the pressure term doesn't appear. By W.K.B. method, we obtain the trianglized profile equations.

In the third section we impose the coherence hypothesis on the set of phases to prevent phases from focusing, formally it leads to the solvability in the category of formal trigonometric series. The coherence hypothesis we used essentially follows the idea of [2], this coherence hypothesis is much more general than that of [6] since the phases needn't to satisfy the eikonal equation. An additional small divisor property implies the compatibility condition for the solvability of the profile equations in the category of smooth quasi-periodic functions. At the end of this section, we deal with the continuity of the singular integrodifferential operator appeared in the reduced system in the second section.

In the fourth section we consider the initial value problem of the compatibility condition. The L^2 estimate makes us be able to obtain the uniqueness and existence of both linear and nonlinear equations. In the fifth section we study the asymptotic validity for the rigorous expansion obtained in the fourth section.

2. Derivation of the Profile Equations by the WKB Method

The system of equations describing the movements of ideal incompressible fluid is

$$\begin{cases} \partial_t V + (V\cdot\nabla)V \doteq -\nabla p \\ \text{div } V = 0 \end{cases} \tag{2.1}$$

where $V(t,x) = (V_1, V_2, \cdots, V_d)$ is the speed vector of the fluid at the point $(t,x) \in [0,T]\times\mathbb{R}^d$, $p(t,x)$ is the scalar pressure at (t,x), $\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_d})$, and div denotes the divergence of a vector. For the system we can pose initial problem.

After partially differentiating to the i-th equation in (2.1) with respect to variable $x_i (1 \le i \le d)$, and using div $V = 0$, we can obtain

$$\Delta p = -\text{tr } (dV)^2 = -\sum_{i,j=1}^{d} \partial_i V_j \partial_j V_i \tag{2.2}$$