J.Partial Diff. Eqs. 9(1996), 169–176
©International Academic Publishers Vol.9, No.2

EXTINCTION AND POSITIVITY FOR THE NON-NEWTONIAN POLYTROPIC FILTRATION EQUATION

Yuan Hongjun
(Department of Mathematics, Jilin University,
Changchun 130023, Jilin, China
and
Department of Physics, Jilin University,
Changchun 130023, Jilin, China)
(Received Sept. 7, 1994; revised Feb. 26, 1995)

Abstract The aim of this paper is to discuss the extinction and positivity for the non-Newtonian polytropic filtration equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right)$$

with $m > 0$, $p > 1$.

Key Words Quasilinear degenerate parabolic equations; extinction and positivity.

Classification 35K65.

1. Introduction

In this paper we consider the non-Newtonian polytropic filtration equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right) \tag{1.1}$$

in Q with initial-boundary conditions

$$u(x,0) = u_0(x), \quad \forall x \in [-1,1]; \quad u(\pm 1, t) = 0, \quad \forall t \in \mathbf{R} \tag{1.2}$$

where $m > 0$, $p > 1$ are given real numbers, and u_0 is a non-zero nonnegative continuous function in $\mathbf{I}$ with $u_0(\pm 1) = 0$, and $Q = \mathbf{I} \times \mathbf{R}$, $\mathbf{I} = [-1,1]$, $\mathbf{R} = (0, +\infty)$.

The equation (1.1) appears in a number of applications to describe the evolution of diffusion processes, in particular the flow of non-Newtonian in a porous medium (cf. [1–3]).

The quasilinear equation (1.1) is degenerate or singular, and therefore has no classical solution in general. We consider its weak solutions.

Definition 1 *A nonnegative function u is said to be a weak solution of* (1.1)–(1.2), *if u satisfies the following conditions:*

$$u \in L^\infty(Q) \cap C(Q), \quad \frac{\partial u^m}{\partial x} \in L^p_{\text{loc}}(Q) \tag{1.3}$$

$$\int_{-1}^{1} u(x,T)\phi(x,T)dx - \int_{-1}^{1} u_0(x)\phi(x,0)dx = \int_0^T \int_{-1}^{1} \left(u\frac{\partial \phi}{\partial t} - \left|\frac{\partial u^m}{\partial x}\right|^{p-2} \frac{\partial u^m}{\partial x} \frac{\partial \phi}{\partial x} \right) dxdt \tag{1.4}$$

for all $T \in (0,\infty)$ *and all* $\phi \in C(0,T;W_0^{1,p}(I))$ *with* $\frac{\partial \phi}{\partial t} \in L^p(Q)$.

The existence, uniqueness and regularity of the solutions of the Cauchy problem (1.1)–(1.2) have been obtained by a number of authors (cf. [4–13]).

In this paper our interest is to investigate the extinction and positivity of solutions. Our main results are the following theorems.

Theorem 1.1 *Let u be a weak solution of* (1.1)–(1.2). *If* $m(p-1) < 1$, *then there exists a time T such that*

$$u(x,t) = 0$$

for all $(x,t) \in \mathbf{I} \times (T,\infty)$.

Remark 1.1 By Theorem 1.1, if $m(p-1) < 1$, then the equation (1.1) has the extinctive property.

Theorem 1.2 *Let u be a weak solution of* (1.1)–(1.2). *If* $m(p-1) > 1$, *then there exists a time T such that*

$$u(x,t) > 0$$

for all $(x,t) \in (-1,1) \times (T,+\infty)$.

Remark 1.2 By Theorem 1.2, if $m(p-1) > 1$, then the equation (1.1) has the positivity and has no extinctive property.

Remark 1.3 Theorem 1.1 and Theorem 1.2 imply that our results are optimal.

Such kind of results have been obtained for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}$$

with $m > 0$ by L.C. Evans in [14], in which the proofs can not be extended to our case. Our approach is different from [14], it is based on some comparison principle.

The proofs of our main results and completed in Section 3 and Section 4. We first prove some fundamental lemmas in Section 2.

2. Fundamental Lemmas

Lemma 2.1 *Let u be a weak solution of* (1.1)–(1.2). *If v satisfies*

$$\frac{\partial v}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial v^m}{\partial x}\right|^{p-2} \frac{\partial v^m}{\partial x} \right)$$