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EXISTENCE AND NONEXISTENCE OF INTERFACES OF WEAK SOLUTIONS FOR NONLINEAR DEGENERATE PARABOLIC SYSTEMS

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Abstract The existence and nonexistence of interfaces of weak solutions for nonlinear degenerate parabolic systems

$$\frac{\partial u_i}{\partial t} - \text{div}\,(|\nabla u|^{p-2}\nabla u_i) = 0, \quad i = 1, 2, \cdots, m$$

with $p > 1$ are studied in this paper.
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1. Introduction

In this paper we consider the parabolic system

$$\frac{\partial u_i}{\partial t} - \text{div}\,(|\nabla u|^{p-2}\nabla u_i) = 0, \quad i = 1, 2, \cdots, m \tag{1.1}$$

in Q with $p > 1$, where $u_i = u_i(x,t)$, $\nabla = \text{grad}_x$, x varies in R^N and $Q = R^N \times (0, +\infty)$.

The parabolic system (1.1) appears in models describing fluids with nonlinear viscosity (see [1]—[3]). It is a nonlinear system, which is degenerate if $p > 2$ or singular if $1 < p < 2$, since the modulus of ellipticity degenerates ($p > 2$) or blows up ($1 < p < 2$) at points where $\nabla u = 0$.

Let the initial condition be

$$u_i(x, 0) = u_{0i}(x), \quad i = 1, 2, \cdots, m \tag{1.2}$$

for $x \in R^N$, and

$$u_{0i} \in C_0^1(R^N), \quad i = 1, 2, \cdots, m \tag{1.3}$$

$$\text{supp } u_{0i} \subset B_{R_0}, \quad i = 1, 2, \cdots, m \tag{1.4}$$

for some $R_0 > 0$, where $B_{R_0} = \{x \in R^N : |x| < R_0\}$.

Definition 1.1 *A vector function* $u = (u_1, u_2, \cdots, u_m)$: $Q \to R^m$ *is said to be a weak solution of the Cauchy problem* (1.1)—(1.2), *if for any* $T > 0$

$$u_i \in L^\infty(Q_T) \text{ and } |\nabla u_i| \in L^p(Q_T); \quad i = 1, 2, \cdots, m \tag{1.5}$$

and u *satisfies the integral identity*

$$\int_{R^N} u_i g_i(x, T)dx + \iint_{Q_T} (-u_i g_{it} + |\nabla u|^{p-2}\nabla u_i \nabla g_i)dxdt = \int_{R^N} u_{0i}(x) g_i(x, 0)dx,$$
$$i = 1, 2, \cdots, m$$

for all vector functions $g = (g_1, g_2, \cdots, g_m)$ *satisfying* $g_i \in C(0, T;\ C_0^1(R^N))$ $(i = 1, 2, \cdots, m)$, *where* $Q_T = R^N \times (0, T)$.

In the case of one equation $(m = 1)$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 1 \tag{1.6}$$

many papers have been devoted to the problem of existence, uniqueness, regularity of solutions, cf. [4]—[8]. In particular, the regularity of interfaces of the weak solutions has been obtained in [9].

In the case of many equations $(m > 1)$, E. Di Benedetto and A. Friedman first have proved the continuity of the gradient of the weak solutions and then have obtained the existence and uniqueness of the Cauchy problem (1.1)—(1.2) in [1]. Further, M. Wiegner has studied the Hölder continuity of the gradient of the weak solutions in [10]. Our interest here is to treat the existence and nonexistence of interfaces of the weak solutions of the Cauchy problem (1.1)—(1.2).

Our main results are the following theorems.

Theorem 1.1 *Assume that* $p > 2$ *and let* $u = (u_1, u_2, \cdots, u_m)$ *be a weak solution of the Cauchy problem* (1.1)—(1.2). *Then the weak solution* u *has the interface, i.e., the set*

$$\{x \in R^N : |u|(x, t) > 0\}$$

is a bounded subset in R^N *for every* $t > 0$, *provided that* (1.4) *holds, where* $|\cdot|$ *is the norm in* R^m.

Theorem 1.2 *Assume that* $1 < p < 2$. *Then there exists* $u_0 \not\equiv 0$ *which satisfies* (1.3) *and* (1.4) *such that the weak solution* $u(x, t)$ *of the Cauchy problem* (1.1)—(1.2) *has no interfaces, i.e., the set*

$$\{x \in R^N : |u|(x, t^*) > 0\}$$

is unbounded in R^N *for some* $t^* \in (0, +\infty)$.

As a direct implication of Theorem 1.1 we have

Proposition 1.1 *The nonlinear degenerate parabolic system* (1.1) *has the finite velocity of the propagation of perturbations under* $p > 2$.