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LIFE-SPAN OF CLASSICAL SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract In this paper, we give a lower bound for the life-span of classical solutions to the Cauchy problem for first order nonlinear hyperbolic systems with small initial data, which is sharp, and give its application to the system of one-dimensional gas dynamics; for the Cauchy problem of the system of one-dimensional gas dynamics with a kind of small oscillatory initial data, we obtain a precise estimate for the life-span of classical solutions.

Key Words Nonlinear hyperbolic system; Cauchy problem; life-span.
Classification 35L60, 35L65.

1. Introduction

Denote by $\widetilde{T}(\varepsilon)$ the life-span considered here, $\widetilde{T}(\varepsilon) = \sup T$ for all $T > 0$ such that there exists a classical solution to the following Cauchy problem for first order nonlinear hyperbolic systems

$$u_t + A(t,x,u)u_x = B(t,x,u) \tag{1}$$

$$t = 0 : u = \varepsilon u_0(x) \tag{2}$$

on $0 \le t \le T$, where $u = (u_1, \cdots, u_n)^T$, $A(t,x,u) = (a_{ij}(t,x,u))$ is an $n \times n$ matrix with C^1 smooth elements $a_{ij}(t,x,u)$ $(i,j = 1,\cdots,n)$ and $B(t,x,u) = (B_1(t,x,u),\cdots,B_n(t,x,u))^T$ is an n-dimensional function vector with C^1 smooth elements $B_i(t,x,u)$ $(i = 1,\cdots,n)$, $\varepsilon > 0$ is a small parameter and $u_0(x)$ is a C^1 function vector with bounded C^1 norm. One of the questions to be discussed is to give a lower bound for the life-span of classical solutions to the Cauchy problem (1)–(2). At the same time, we give an example to show that our estimate is sharp. On the other hand, we furthermore discuss the life-span of classical solution to the following Cauchy problem for the system of one-dimensional gas dynamics:

$$\begin{cases} \tau_t - u_x = 0 \\ u_t + p_x = 0 \\ S_t = 0 \end{cases} \tag{3}$$

$$t=0: u=\overline{u}+\varepsilon u_0(x),\ p=\overline{p}+\varepsilon \overline{p}_0,\ S=\varepsilon S_0\Big(\frac{x}{\varepsilon^a}\Big) \tag{4}$$

where τ, u, p and S are respectively the specific volume, velocity, pressure and entropy of the gas, $\overline{p}>0$, $\overline{u}$ are constants, $u_0(x), p_0(x), S_0(x)$ are C^1 functions with compact supports, $\varepsilon>0$ is a small parameter, $0\le a<1$ is a constant, and we give a precise estimate for the life-span.

It is well-known that when $A(t,x,u)=A(u)$, $B(t,x,u)\equiv 0$, a number of results have been obtained both on global existence and on blow-up of solutions (cf. [1]–[4]). F. John [1] and T.P. Liu [2] showed that blow-up always occurs in the genuinely nonlinear case for small initial data of compact support, and gave an upper bound estimate for the life-span. In Chapter I of [3], L. Hörmander determined the time of blow-up asymptotically, and gave a self-contained and somewhat simplified exposition of these methods. By introducing the concept of weak linear degeneracy, Li Tatsien, Zhou Yi & Kong Dexing [4] gave a complete result on the global existence and the life-span of C^1 solution to the Cauchy problem for general homogeneous quasilinear hyperbolic systems with small initial data. For the Cauchy problem (3)–(4), when $a=0$, it was proved in [2] and [4] that there exists a small $\varepsilon_0>0$ such that for any given $\varepsilon\in(0,\varepsilon_0]$ the C^1 solution of the Cauchy problem (3)–(4) must blow up at a finite time, the life-span $\widetilde{T}(\varepsilon)$ is $O(\varepsilon^{-1})$.

Our approach is based on some formulas on the decomposition of waves, we give a brief derivation of these formulas in Section 2; in Section 3, we give the lower bound for the life-span of classical solutions to the Cauchy problem (1)–(2); in Section 4, we give an application of the result abovementioned to the system of one-dimensional gas dynamics; in Section 5, we give the precise estimate for the life-span of classical solutions to the Cauchy problem (3)–(4).

2. Formal Theory of the Differential Equations

By the definition of hyperbolicity, for any given $(t,x,u)\in\mathbf{R}^+\times\mathbf{R}\times\mathbf{D}$ (where $\mathbf{D}$ is the considerable domain of u),

1) $A(t,x,u)$ has n real eigenvalues $\lambda_1(t,x,u),\cdots,\lambda_n(t,x,u)$;

2) $A(t,x,u)$ is diagonalizable, i.e., there exists a complete set of left (resp. right) eigenvectors. Let $l_i(t,x,u)=(l_{i1}(t,x,u),\cdots,l_{in}(t,x,u))$ (resp. $r_i(t,x,u)=(r_{i1}(t,x,u),\cdots,r_{in}(t,x,u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(t,x,u)$ $(i=1,\cdots,n)$:

$$\begin{aligned}&l_i(t,x,u)A(t,x,u)=\lambda_i(t,x,u)l_i(t,x,u)\\ &\qquad(\text{resp. } A(t,x,u)r_i(t,x,u)=\lambda_i(t,x,u)r_i(t,x,u))\end{aligned} \tag{5}$$

we have

$$\det|l_{ij}(t,x,u)|\neq 0 \quad (\text{equivalently, } \det|r_{ij}(t,x,u)|\neq 0) \tag{6}$$