J.Partial Diff. Eqs. 9(1996), 263–276
©International Academic Publishers
Vol.9, No.3

TWO-SCALE CONVERGENCE AND HOMOGENIZATION FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

Zhang Xingyou
(Dept. of Applied Math., Chongqing University, Chongqing 630044, China)
(Received Oct. 31, 1994; revised Sept. 25, 1995)

Abstract By use of Fourier analysis techniques, we obtain some new properties of the almost-periodic functions and extend the two-scale convergence method in the homogenization theory to the case of almost-periodic oscillations. Then, we use some new techniques to study the homogenization for quasilinear elliptic equations with almost-periodic coefficients:

$$\operatorname{div} a(x, x/\varepsilon, u, Du) = f(x) \quad \text{in } \Omega$$

and obtain the weak convergence and corrector results.
Key Words Partial differential equations; homogenization; weak convergence; almost-periodic; two-scale convergence.
Classification 35B40, 41A35.

1. Introduction

The homogenization theory involves the study of asymptotic behaviour of the solution u_ε of the equation $L_\varepsilon u_\varepsilon = f$ as $\varepsilon \to 0$. It has wide applications in the study of properties of composite material, and there are lots of papers devoted to it (see [1], [2] and the references there).

De Giorgi's variational convergence method and L. Tartar's energy method are two main classic approaches in the homogenization theory, however, they have not utilized fully the periodicity in homogenization. G. Nguetseng [3] proposed a new method (the so-called *two-scale convergence method*) which exploits fully the periodicity in homogenization and has extensive applications. G. Allaire [4] developed the two-scale convergence method and studied its applications to several problems, meanwhile, he pointed out that this method applied only to pure periodic oscillation, and we did not know whether it applied to almost-periodic oscillations [4, p1484].

Here, we will answer affirmatively this problem. By use of some Fourier analysis techniques, we get some new properties of almost-periodic functions, and establish the corresponding two-scale convergence theory. Then, we study the asymptotic behavior of the solution $u_\varepsilon \in H_0^1(\Omega)$ to the problem

$$\operatorname{div} a\Big(x, \frac{x}{\varepsilon}, u, Du\Big) = f(x) \quad \text{in } \Omega$$

as $\varepsilon \to 0$, where $a(x, y, u, Du)$ is almost-periodic with respect to y. Many papers are devoted to the homogenization of this problem with pure periodic oscillation (see, e.g. [1], [2], [5] etc.), G. Allaire [4] has studied it by two-scale convergence method for $a\Big(x, \dfrac{x}{\varepsilon}, u, p\Big)$ independent of u. Here we use some new techniques to overcome the difficulty caused by the dependence of $a\Big(x, \dfrac{x}{\varepsilon}, u, p\Big)$ on u, and obtain the weak convergence and corrector results.

2. Almost-periodic Functions and Two-scale Convergence

Definition 2.1 *A function $f : R^n \to R^1$ is uniformly almost-periodic, if there are sequences of numbers a_k and of vectors $\mathbf{r}_k$ such that $\sum_{n=1}^{\infty} |a_k| < +\infty$ and $f(x) = \sum_{k=1}^{\infty} (a_k e^{i\mathbf{r}_k x} + \overline{a}_k e^{-i\mathbf{r}_k x})$, where $\overline{a}_k$ denotes the conjugate complex of a_k, we denote it by $f \in UAP(R^n)$.*

Proposition 2.1[6] *If $f \in UAP(R^n)$, then for any open set $\Omega \subset R^n$, the limit*

$$\lim_{s\to\infty} \frac{1}{|s\Omega|} \int_{s\Omega} f(x)dx$$

exists, is finite and independent of Ω.

Let Q be the unit cube $\Big(-\dfrac{1}{2}, \dfrac{1}{2}\Big)^n \subset R^n$, then we have

Corollary 2.1 If $f \in UAP(R^n)$, then the limit

$$|f|_{B_2} = \lim_{s\to\infty} \Big(\frac{1}{|sQ|} \int_{sQ} |f(x)|^2 dx\Big)^{\frac{1}{2}} \tag{2.1}$$

exists and is finite.

It is easy to verify that $|\cdot|_{B_2}$ defined by (2.1) is a norm on $UAP(R^n)$.

Proposition 2.2[7] If $f \in UAP(R^n)$, η is a mollifier, then $f_\eta(x) = \int_{R^n} \eta(y) f(x - y)dy \in UAP(R^n)$.

Definition 2.2 *Let $X_2(R^n)$ be the completion space of $UAP(R^n)$ with respect to the norm $|\cdot|_{B_2}$, $H \equiv \{u \in X_2(R^n) : |u|_{B_2} = 0\}$, $B_2(R^n) \equiv X_2(R^n)/H$.*

It is easy to prove the following Lemma 2.1 and Proposition 2.3.