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UNIQUENESS OF GENERALIZED SOLUTIONS FOR A QUASILINEAR DEGENERATE PARABOLIC SYSTEM

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Abstract In this paper we study the uniqueness of generalized solutions for a class of quasilinear degenerate parabolic systems arising from dynamics of biological groups. The results obtained give an answer to a problem posed by A.S. Kalashnikov [1].

Key Words Uniqueness; weak solution; quasilinear degenerate parabolic system.
Classification 35K55, 35K65.

In this paper we consider a quasilinear degenerate parabolic system of the form

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i^{m_i} + b_i u_1^{p_i} u_2^{q_i} \tag{1}$$

in $Q_T = R^T \times (0,T)$ with the initial condition

$$u_i(x,0) = u_{0i}(x) \tag{2}$$

for $x \in R^N$, where $m_i \geq 1, p_i \geq 1, q_i \geq 1, T > 0, a_i > 0$, b_i are given real numbers and $u_{0i} (i = 1,2)$ are bounded measurable functions in R^N.

The system (1) arises from modeling interacting evolution of two biological groups with densities u_1, u_2 (see [1]).

Definition *A vector function (u_1, u_2) with $u_i \in L^\infty(Q_T)$ and $u_i \geq 0$ $(i = 1,2)$ is a generalized solution of (1)–(2), if (u_1, u_2) satisfies*

$$\iint_{Q_T} (u_i \varphi_{it} + a_i u_i^{m_i} \Delta \varphi_i + b_i u_1^{p_i} u_2^{q_i} \varphi_i)\, dxdt = 0 \tag{3}$$

for all $\varphi_i \in C_0^\infty(Q_T)$, $i = 1,2$;

$$\lim_{t \to 0} \int_{R^N} \psi_i (u_i(x,t) - u_{0i}(x)) dx = 0 \tag{4}$$

for all $\psi_i \in C_0^\infty(R^N)$, $i = 1,2$.

A.S. Kalashnikov first studied the system (1) and proved the existence of generalized solutions to the Cauchy problem (1)–(2) (see [1]). However, he was not able to solve the problem of uniqueness, and put forward as an open problem in the paper [1]. Afterwards, A.S. Kalashnikov mentioned this problem again at a symposium [2].

In this paper, we attempt to give an answer to this problem. The main result obtained is the following theorem.

Theorem *Let the vector functions (u_1, u_2) and (v_1, v_2) be two generalized solutions of the Cauchy problem* (1)–(2). *Then*

$$u_i(x,t) = v_i(x,t), \quad i = 1,2$$

for a.e. $(x,t) \in Q_T$.

Here the uniqueness is proved for the cases $p_i \geq 1$ and $q_i \geq 1$ $(i = 1,2)$. The following example shows that these conditions can not be removed in general: both

$$w = 0 \quad \text{and} \quad w = [(1-p)t]^{\frac{1}{1-p}}$$

are generalized solutions of the equation

$$w_t = \Delta w^m + w^p \tag{5}$$

in $Q_T = R^N \times (0,T)$ with the initial condition

$$w(x,0) = 0 \tag{6}$$

on R^N, where $m > 1$ and $1 > p > 0$. In case $b_i < 0$ $(i = 1,2)$ the uniqueness seems to be true even if $p_i < 1$ and $q_i < 1$ $(i = 1,2)$. But we are not able to prove yet.

The result (Theorem) can be extended to more general systems of the form

$$\frac{\partial u_i}{\partial t} = \Delta A_i(u_i) + B_i(u_1, u_2, \cdots, u_n), \quad i = 1,2,\cdots,n$$

in Q_T, where $A_i : R^1 \to R^1$, $B_i : R^n \to R^1$ $(i = 1,2,\cdots,n)$ are locally Lipschitz continuous, respectively.

The same problem for more general systems with convection term has been studied by one of the authours and his colleague and the uniqueness of BV solutions has been proved (see [3]).

In order to prove the theorem we define

$$A_i(x,t) = \begin{cases} a_i \cdot \dfrac{u_i^{m_i}(x,t) - v_i^{m_i}(x,t)}{u_i(x,t) - v_i(x,t)} & \text{if } u_i(x,t) \neq v_i(x,t) \\ 0, & \text{otherwise} \end{cases}$$

$$A_{i,\varepsilon}(x,t) = A_i(x,t) + \varepsilon, \ i = 1,2$$

$$A_{i,\varepsilon,\rho}(x,t) = (A_{i,\varepsilon} * J_\rho)(x,t), \ i = 1,2$$

for all $(x,t) \in Q_T$, where

$$0 < \rho < 1, \quad J_\rho \in C^\infty(R^{N+1}), \ \int_{R^{N+1}} J_\rho = 1$$