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ANALYSIS OF AN INTEGRO-DIFFERENTIAL EQUATION ARISING FROM MODELLING OF FLOWS WITH FADING MEMORY THROUGH FISSURED MEDIA

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Abstract An analysis of an integro-differential equation with a convolution term is given. Such equations arise in modelling of flows through fissured media, and these integral terms account for fading memory effects exhibited by the flow. We proposed a convergent semi-discrete approximation of the convolution term with a possibly singular kernel. The approximation scheme leads to the existence/uniqueness result for the problem and has strongly favorable numerical aspects.

Key Words Integro-partial differential equations; existence and uniqueness of solutions; convolution integrals.

Classification 35K99, 45K05, 35A35.

1. Introduction

Models of microstructure phenomena have recently attracted much interest. This is related to the appearance of new techniques of modelling like homogenization methods (see [1, 2]) and new achievements in numerical methods, especially recent developments in parallel computing. The models of flows through fissured media are examples of microstructure models which require nonstandard techniques for their analysis and approximation.

Fissured media are porous media of hierarchical geometrical structure. Below we are concerned with a model of flow through fissured medium proposed and analysed by Hornung and Showalter in [3], see also [4], which is derived by homogenization and takes into account the fading memory effects exhibited by the flow. The equation we deal with has the form

$$u_t(x,t) + \int_0^t u_t(x,s)\tau(t-s)ds - \nabla \cdot (D \nabla u(x,t)) = f(x,t),\ (x,t) \in \Omega \times I \tag{1}$$

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times I \tag{2}$$

$$u(x,0) = u_0(x),\ x \in \Omega \tag{3}$$

where an open bounded set $\Omega \subset R^d$, $d = 2$ or $d = 3$, with boundary $\partial\Omega$ is the domain of the flow. $I = (0,T)$, $T > 0$ is the time interval, and D is the coefficient tensor (possibly dependent on space variable). The convolution kernel $\tau(\cdot)$ is related to the microscopic properties of the domain of the flow and is by definition (see [3, 5]) singular at $t = 0$

but L^1 integrable. For example, for some particular microscopic geometric structure of the medium the convolution kernel is given by

$$\tau(t) = c_0 \sum_{k=1}^{\infty} e^{-c_1 k^2 t}$$

where the coefficients c_0, c_1 depend on the properties of the medium (see [5]).

We note that in the case $\tau \equiv 0$ Equation (1) reduces to the standard linear diffusion equation. The same reduction is valid if τ is given as the Dirac measure concentrated at $t = 0$. Both of these trivial cases describe diffusion with the postulated instantaneous propagation of changes in the values of variables. The nature of the flow in fissured media requires $\tau \not\equiv 0$, $\tau \not\equiv \delta$, i.e., the presence of nontrivial memory terms in the model. In the case considered in this paper those memory terms admit the convolution representation with kernels unbounded at the origin. It has been observed that the models with memory terms with bounded kernels lead to a less accurate description of the dynamics of the flow than is the case treated in this paper, i.e., the case of singular kernels (see [3, 6]). We mention also the work of Arbogast and Douglas in [7, 8] who use a different modelling approach in order to describe the memory effects. Their work however can be imbedded in a common framework together with the model above by means of an application of a generalized convolution operator proposed in [5].

The models corresponding to the case when (1) reduces to the linear diffusion equation yield to the standard analytic and numerical treatment. In this paper we are focussed on the proper treatment of the memory terms with nontrivial kernels. We propose a convergent approximation scheme which combined with the method of lines (also known as Rothe method) leads to the existence/uniqueness result for the model. The advantage of our method with respect to the one applied in [3] is in the less restrictive assumptions on the elliptic part of the problem. Also, our approximation algorithm has a strong numerical aspect as it provides a tool for numerical treatment of integral terms with singular kernels (see [5, 9] in this direction). Such integral terms appear frequently in viscoelasticity theory ([10] and references given there), theory of phenomena with memory [11] as well as in homogenization limits of scalar conservation laws [12]. The method presented below has many potential analytical and numerical applications.

The paper is organized as follows. In Section 2 we analyse properties of the convolution term under relatively weak assumptions on the convolution kernel. In Section 3 we define a method of approximation of the convolution term and formulate basic technical lemmas. In Section 4 we prove the main result of this paper on the existence of a unique weak solution to the problem (1)–(3).

2. Convolution Operator

In this section we recall and study properties of the convolution operator $L_\tau : L^2(I) \mapsto L^2(I)$ or, in general $L_\tau : L^2(I;\mathcal{H}) \mapsto L^2(I;\mathcal{H})$ with $\mathcal{H}$ a Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$). Let $\tau \in L^1(I)$ be fixed. The operator L_τ is defined by

$$(L_\tau v)(t) = (\tau \star v)(t) = \int_0^t \tau(t-s)v(s)ds, \quad v \in L^2(I),\ t \in I$$