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ON INHOMOGENEOUS GBBM EQUATIONS

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Abstract In this paper we consider the Cauchy problem and the initial boundary value (IBV) problem for the inhomogeneous GBBM equations. For any bounded or unbounded smooth domain, the existence and uniqueness of global strong solution for the Cauchy problem and IBV problem for the inhomogeneous GBBM equations in $W^{2,p}(\Omega)$ are established by using Banach fixed point theorem and some *a priori* estimates. These results have improved the known results even in the case of GBBM equation. Meanwhile, we also discuss the regularity of the strong solution and the system of inhomogeneous GBBM equations.

Key Words Inhomogeneous GBBM equation; Cauchy problem; IBV problem; strong solution.

Classification 35Q20.

1. Introduction and Main Results

In this paper, we discuss the following inhomogeneous Generalized Benjamin-Bona-Mahony equation

$$u_t - \Delta u_t - \mathrm{div}\phi(u) = f(u) \tag{1.1}$$

which satisfies the IBV conditions

$$\begin{cases} u(x,0) = u_0(x), & x \in \Omega \subset \mathbb{R}^n \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty) \end{cases} \tag{1.2}$$

or Cauchy condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \tag{1.3}$$

Here Ω is a bounded or unbounded smooth domain in $\mathbb{R}^n$, $n \geq 1$. $\phi(u) = (\phi_1(u), \cdots, \phi_n(u))$ is a nonlinear vector function, $f(u)$ is a nonlinear function.

The classical **BBM** equation

$$u_t - u_{xxt} + \left(u + \frac{1}{2}u^2\right)_x = 0 \tag{1.4}$$

and the following GBBM equation in arbitrary space dimensions

$$u_t - \Delta u_t + \mathrm{div}\phi(u) = 0 \tag{1.5}$$

have been quite extensively studied by many authors [1–8]. Their main results can be stated as the following:

(i) For any bounded smooth domain Ω and $\max\left(1,\frac{n}{2}\right) < p < \infty$, there is a unique global strong solution for the problem (1.5) (1.2) in $W^{2,p}(\Omega)$; for any unbounded smooth domain Ω or $\Omega = \mathbb{R}^n$, $\max\left(1,\frac{n}{2}\right) < p < n$, there exists a unique global strong solution for the problem (1.5) (1.2) or (1.5) (1.3) [2, 4, 7–9].

(ii) For $p \le \frac{n}{2}$ if the components $\phi'(u)$ are polynomials of degree $q \le \frac{p}{n-2p}\left(p < \frac{n}{2}\right)$, there is a unique local strong solution for the problem (1.5) (1.2) or (1.5) (1.3). In particular, there exists a unique global strong solution for the problem (1.5) (1.2) or (1.5) (1.3) in $W^{2,2}(\Omega)$ [3,5]. This paper establishes the existence and uniqueness of global strong solution for the IBV problem and Cauchy problem of inhomogeneous GBBM equation in $W^{2,p}(\Omega)$ in the case of $\max\left(1,\frac{n}{2}\right) < p < \infty$, where $f(u)$ and $\phi(u)$ satisfy subcritical and critical growth. Meanwhile, the regularity of the strong solution and the system of inhomogeneous GBBM equation are discussed. Our results have improved the known results even in the case of GBBM equation.

Let $\max\left(1,\frac{n}{2}\right) < p < \infty$, we define an operator $A_p = (I - \Delta)$ with domain $D(A_p) = W^{2,p}(\Omega) \cap W_0^{2,p}(\Omega)$, where I denotes identity in $L^p(\Omega)$ and Δ denotes the Laplace operator. It is easy to see that A_p has a bounded inverse operator A_p^{-1} form $L^p(\Omega)$ into $D(A_p)$. The $D(A_p)$ becomes a Banach space with respect to the graph norm

$$\|A_p u\|_p = \|(I-\Delta)u\|_p \tag{1.6}$$

It is well known that the graph norm is equivalent to $W^{2,p}(\Omega)$ norm. Hence, we shall use $W^{2,p}$ norm $\|\cdot\|_{2,p}$ on the Banach space $D(A_p)$ in this paper.

Lemma 1.1 *Let Ω be a bounded or unbounded smooth domain in $\mathbb{R}^n$, $\max\left(1,\frac{n}{2}\right) < p < \infty$, assume that*

(1) $\phi \in C^2(\mathbb{R}, \mathbb{R}^n)$, $\phi(0) = 0$

(2) $f \in C^1(\mathbb{R}, \mathbb{R})$; If Ω is an unbounded smooth domain or $\Omega = \mathbb{R}^n$, we further assume that $f(0) = 0$.

Then under the condition of $u(t) \in C([0,\infty); D(A_p))$, we have $\nabla\phi(u) \in C([0,T]; L^p(\Omega))$, $A_p^{-1} f(u) \in C([0,T]; D(A_p))$.

Proof By Sobolev embedding theorem we know that $D(A_p) \hookrightarrow L^\infty(\Omega)$. Using the definition of divergence we have

$$\mathrm{div}\phi(u) = \nabla \cdot \phi(u) = \phi'(u) \nabla u \in L^p(\Omega), \quad \forall u \in D(A_p) \tag{1.7}$$

Hence, $\phi(u)$ can be regarded as an operator from $D(A_p)$ to $(W_0^{1,p}(\Omega))^n$. Noting that (2) and Sobolev imbedding theorem we always have $f(u) \in L^p(\Omega)$.

Definition *Let $T > 0$, $\max\left(1,\frac{n}{2}\right) < p < \infty$, $u_0(x) \in D(A_p)$. We call the function $u(x,t) \in C([0,\infty); D(A_p))$ a strong solution of the problem* (1.1) (1.2) *or* (1.1)