J.Partial Diff. Eqs. 8(1995), 321–332
©International Academic Publishers Vol.8, No.4

ON SMOOTH SOLUTION FOR A NONLINEAR 5TH ORDER EQUATION OF KDV TYPE

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(Received July 9, 1993; revised June 11, 1994)

Abstract The existence of global smooth solutions for a nonlinear 5th order equation of KdV type with the periodic boundary condition and initial value condition is proved, we also get the local smooth characterization of the solution for the initial value problem.

Key Words Smooth solution; local smooth characterization.
Classification 35Q35.

1. Introduction

Many nonlinear higher order equations of KdV type have been proposed in physical problem. In [1] Benney considers a model partial differential equation of the form

$$u_t + Lu = Nu \tag{1}$$

Here L is a linear operator which consists of only x odd ordered differentiations, and N is some nonlinear operator which can be the following simple acceptable forms:

$$(A)uu_x; \quad (B)uu_{xxx} + 2u_x u_{xx}; \quad (C)u_x u_{x^4} + 5u_{xx}u_{xxx} \tag{2}$$

or any linear combination of cases (A), (B) and (C). He studies the interactions between short and long waves using (1) and gets fine results. In [2] in order to more exactly represent the behaviour of the nonlinear lattice in the limit of long waves Lisher proposes the following equation

$$v_t + \Big(\frac{\partial F(v, v_x)}{\partial v}\Big)_x - \Big(\frac{\partial F(v, v_x)}{\partial v_x}\Big)_{xx} + \epsilon v_{x^3} + \eta v_{x^5} = 0 \tag{3}$$

where

$$F(v, v_x) = \frac{1}{6}v^3 + \frac{\gamma}{12}v^4 - \frac{\rho}{4}vv_x^2 + \frac{\delta}{2}v^2v_x^2 \tag{4}$$

$$\Big(\frac{\partial F(v, v_x)}{\partial v}\Big)_x = \Big(\frac{\partial F(q, r)}{\partial q}\Big|_{(q,r)=(v,v_x)}\Big)_x$$

$$\left(\frac{\partial F(v,v_x)}{\partial v_x}\right)_{xx} = \left(\frac{\partial F(q,r)}{\partial r}\Big|_{(q,r)=(v,v_x)}\right)_{xx}$$

all of $\epsilon, \eta, \gamma, \rho, \delta$ are constants. In [3] Lax extends KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5}$$

to higher orders and gets 5th order KdV equation

$$u_t + 30u^2u_x + 10(2u_xu_{xx} + uu_{xxx}) + u_{x^5} = 0 \tag{6}$$

For Equation (6) infinitely many conservation laws and inverse scattering form have been found. In [4–7] the way of extending KdV equation (5) to higher order equations which possesses infinitely many conservation laws, Backlund transformation and inverse scattering forms, has been studied; and soliton solutions for the resulting equations have also been studied. The existence of weak solution for a generalized higher order nonlinear system of equations of KdV type with periodic boundary condition and initial value condition has been obtained by Zhou and Guo in [8].

In this paper, we consider the existence of smooth solution for the following equation

$$Lu = u_t + \left(\frac{\partial F(u)}{\partial u}\right)_x + \left(\frac{\partial G(u,u_x)}{\partial u}\right)_x - \left(\frac{\partial G(u,u_x)}{\partial u_x}\right)_{xx} + u_{x^5} = 0 \tag{7}$$

with periodic boundary condition

$$\begin{cases} u(x,t) = u(x+2D,t), \quad D > 0 \\ u(x,0) = \psi(x), \ \psi(x) = \psi(x+2D) \end{cases} \tag{8}$$

and the existence of smooth solution for Equation (7) with initial value condition

$$u(x,0) = \phi(x) \tag{9}$$

where

$$\begin{cases} G(u,u_x) = N_1 + N_2 + N_3 + N_4, \\ N_1 = a_1uu_x^2, \quad N_2 = a_2u^2u_x^2, \quad N_3 = a_3u^3u_x^2, \quad N_4 = a_4u_x^3 \end{cases} \tag{10}$$

$$\left(\frac{\partial F(u)}{\partial u}\right)_x = \left(\frac{\partial F(q)}{\partial q}\Big|_{q=u}\right)_x$$

$$\left(\frac{\partial G(u,u_x)}{\partial u}\right)_x = \left(\frac{\partial G(q,r)}{\partial q}\Big|_{(q,r)=(u,u_x)}\right)_x$$

$$\left(\frac{\partial G(u,u_x)}{\partial u_x}\right)_{xx} = \left(\frac{\partial G(q,r)}{\partial r}\Big|_{(q,r)=(u,u_x)}\right)_{xx}$$

a_i $(i = 1,2,3,4)$ are constants. Suppose that $F(u), \psi, \phi$ satisfy the following conditions

$$\begin{cases} (1) \quad F(\xi) \in C^{s+2}(R), \ |F''(\xi)| \le A_1(1+|\xi|^7) \\ \qquad F'(0) = F(0) = 0 \\ (2) \quad \psi \in H^s(-D,D), \quad \phi \in H^s(R), \ s \ge 5 \end{cases} \tag{11}$$