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ON THE GEOMETRIC MEASURE OF NODAL SETS OF SOLUTIONS*

Han Qing and Lin Fanghua
(Courant Inst.of Math.Sci.251 Mercer st New York, NY10012 U.S.A.)

Dedicated to Professor Guo Zhurui for his 70th Birthday

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Abstract We present some general methods for the estimation of the local Hausdorff measure of nodal sets of solutions to elliptic and parabolic equations. Our main results (Theorems 3.1 and 4.1) improve previous results of Lin Fanghua in [1].

Key Words Hausdorff measure, nodal set, elliptic and parabolic equations
Classification 35J15, 35K10

Introduction

The aim of this article is to give some general methods of estimating the local Hausdorff measure of nodal sets (or level sets in general) of solutions to elliptic and parabolic equations.

The first method is based on some compactness, approximation and covering arguments. For this reason we require, first of all, those equations under consideration possess certain compactness such as: the uniform ellipticity (or parabolicity), the leading coefficients are uniformly continuous and the lower order terms are uniformly under control, etc. For the same reason we also require that those solutions under consideration have some compactness such as the L^2-doubling condition (or the natural L^2-growth hypothesis), see, e.g., [2] and [3]. Under these assumptions we can show that the Hausdorff measures (of appropriate dimensions) of nodal sets are locally finite.

The basic idea in this method is rather simple. One decomposes a nodal set inside a ball into a "good" part, G, and a "bad" part, B. On G the gradient of the solution is relatively large so that G is smooth and locally can easily be approximated by nodal sets of solution of elliptic (or parabolic) equations with constant coefficients. Since the Hausdorff measure of nodal sets of solutions to elliptic or parabolic equations with constant coefficients can be estimated as in [1], we thus can control the corresponding Hausdorff measure of G. For the "bad" part B, we show it can be covered by a family

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of balls $\{B_{r_i}(x_i)\}$ so that $\sum_i r_i^{n-1}$ is sufficiently small. Then we repeat the above arguments on each ball $B_i = B_{r_i}(x_i)$ to obtain a desired estimate. Since the proof involves compactness arguments, we lose an explicit estimate such as that obtained in [2] and [3]. Also since we use the fact that the singular set (that is the set on which both the solution and its gradient vanish) has definitely lower dimensions than that of the nodal set, the argument above does not apply to estimating some suitable slices of nodal sets of solutions to elliptic and parabolic equations.

The second method is particularly useful in estimating some suitable slices of level sets of solutions to parabolic and elliptic equations. However, for a technical reason we need to make an additional hypothesis on the coefficients of equations. To be specific, we assume that these coefficients are smooth. Under this additional hypothesis, we can use some integral geometry estimate for harmonic or caloric functions to estimate the geometric measure of suitable slices of nodal sets. This is probably the first such estimate for smooth coefficient operators (cf.[1]).

The study of the geometric measure and global behavior of nodal sets of solutions to elliptic and parabolic equations has become an interesting topic recently. In a series of papers, Donnelly and Fefferman [4] [5] [6] studied the nodal sets of eigenfunctions of the Laplacian on a compact Riemannian manifold. They found both upper and lower bounds on Hausdorff measures of nodal sets in the case that the given manifold is analytic. We refer to [6] and [7] for the case that the metric is smooth and the manifold is 2-dimensional. On the other hand, Hardt and Simon [2] and the authors [3] studied nodal sets of solutions to a general class of elliptic and parabolic equations of second order without analyticity hypothesis. Some explicit estimates, though they may not be optimal, on local Hausdorff measures of nodal sets under a natural growth assumption on solutions have been proven. In [1], one of the authors obtained an optimal upper bound of the Hausdorff measure of nodal sets of solutions for second order elliptic equations with analytic coefficients and parabolic equations with time independent analytic coefficients. We should also point out some interesting questions concerning point vertices in the Ginsburg-Landau model for problems related to super conductivity, see [8] for details.

We are very happy to dedicate this paper to Professor Guo Z.R. on the occasion of his 70th birthday. Though the paper is on the subject of partial differential equations, some key arguments involved are those approximating solutions by polynomials, compactness, covering and integral geometric estimates, etc. Professor Guo has made important contributions to the latter subjects.

The paper is written as follows. In order to simplify the presentation, we should mainly discuss solutions to elliptic equations. We shall, however, add various remarks in the paper to show how these results can easily be generalized to the case of parabolic equations. In the first section, we introduce classes of operators and solutions which satisfy some nice compactness property. We shall also discuss some important facts