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GLOBALLY SMOOTH SOLUTIONS TO AN INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEM ARISING IN CHEMICAL ENGINEERING

Hu Jiaxin, Zhao Huijiang and Zhu Changjiang
(Inst. of Math. Scis., Academia Sinica, P. O. Box 71007, Wuhan 430071, China)
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Abstract In this paper we have obtained the existence of globally smooth solutions to an inhomogeneous nonstrictly hyperbolic system $u_t - (v(1-u))_x = 0$, $v_t + \left(\frac{1}{2}v^2 - c_0 u\right)_x = f(u,v)$ by employing the characteristic method and the fixed-point theorem in Banach spaces.
Key Words Nonstrictly hyperbolic system; characteristic curve; fixed point theorem.
Classification 35L.

1. Introduction

The purpose of this paper is to study the Cauchy problem for an inhomogeneous quasilinear hyperbolic system

$$\begin{cases} u_t - (v(1-u))_x = 0 \\ v_t + \left(\frac{1}{2}v^2 - c_0 u\right)_x = f(u,v) \end{cases} \quad (x,t) \in \boldsymbol{R} \times \boldsymbol{R}_+ \tag{1.1}$$

This model arises in chemical engineering and is treated as a nonstrictly hyperbolic system. Here $f(u,v)$ is a C^2-smooth mapping from $\boldsymbol{R}^2$ to $\boldsymbol{R}$ and c_0 is a positive constant. Without loss of generality we assume that $c_0 = 1$ in the following context. For such a system, the two eigenvalues are

$$\lambda_1 = v - (1-u)^{\frac{1}{2}}, \quad \lambda_2 = v + (1-u)^{\frac{1}{2}}$$

which coalesce on the line $u = 1$ in the (u,v)-plane. Thus the system is not strictly hyperbolic over the domain $\{(u,v) : -\infty < u \leq 1, -\infty < v < +\infty\}$. The corresponding Riemann invariants are respectively:

$$w(u,v) = v + 2(1-u)^{\frac{1}{2}}, \quad z(u,v) = v - 2(1-u)^{\frac{1}{2}}$$

We recall that for strictly hyperbolic systems, the global existence of smooth solutions to the Cauchy problem has been obtained in [1]–[4] by applying various methods. These results are quite perfect. However, to our knowledge, very few results have been acquired about the existence of globally smooth solutions to nonstrictly hyperbolic systems of quasilinear equations. On the other hand, one turns to consider globally continuous solutions to nonstrictly hyperbolic settings. In this regard, we refer the reader to the work of [5]–[7]. In this paper our interest is mainly in globally smooth solutions to (1.1) with initial values

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad x \in \boldsymbol{R} \tag{1.2}$$

by employing the characteristic method and the fixed-point theorem in Banach spaces. This technique agrees, in spirits, with the one used in the arguments of [1] and [2]. We believe that the technique can also be applied to many other cases.

It is well-known that smooth solutions, in general, may not exist globally to the Cauchy problem of hyperbolic systems of first order quasilinear equations even if initials are sufficiently smooth. As a matter of fact, shock waves are generally involved in solutions as time evolves. Of course, in this article there are no shock waves but rarefaction waves in solutions when initial data satisfy certain mandatory restrictions. Thus it is required to investigate generalized (weak) solutions to quasilinear hyperbolic equations. The recently-developed theory of compensated compactness has proven to be a powerful tool in establishing the existence theorem of generalized (weak) solutions to quasilinear hyperbolic conservation laws for large initial data. Nevertheless, no framework has been established on proving the compactness of approximate solutions for nonstrictly hyperbolic system, which contrast sharply with strictly hyperbolic systems[8], although existence results have been obtained for some special nonstrictly hyperbolic conservation laws[9]–[14]. This is the reason why we want to study globally smooth solutions to (1.1) and (1.2).

2. Smooth Solutions

Our goal is to prove the existence of globally smooth solutions to (1.1) and (1.2). For this purpose we first study the following initial value problem

$$\begin{cases} w_t + \lambda_2(w,z)w_x = g(w,z) \\ z_t + \lambda_1(w,z)z_x = g(w,z) \end{cases} \quad (x,t) \in \boldsymbol{R} \times \boldsymbol{R}_+ \tag{2.1}$$

and

$$(w(x,t), z(x,t))|_{t=0} = (w_0(x), z_0(x)), \quad x \in \boldsymbol{R} \tag{2.2}$$

where $g(w,z) = f\left(1 - \frac{1}{16}(w-z)^2,\ \frac{1}{2}(w+z)\right)$ and $\lambda_2(w,z) = \frac{3}{4}w + \frac{1}{4}z$, $\lambda_1(w,z) = \frac{1}{4}w + \frac{3}{4}z$. The initial values are defined by

$$w_0(x) = v_0(x) + 2(1 - u_0(x))^{\frac{1}{2}}$$