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ON THE BLOW-UP FOR QUASILINEAR PARABOLIC EQUATIONS

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Abstract This article is concerned with the position of blow-up points, blow up rate and an isoperimetric problem for the equation $u_t = \Delta u^m + u^p$ $(p > m \geq 1)$ in a convex bounded domain.

Key Words Blow-up points; blow-up rate; isoperimetric problem
Classification 35K

1. Introduction

Recently there is an extensive literature on properties of solutions for nonlinear parabolic equations, such as blow-up, quenching, dead core and extinction, see [1-4]. Especially more are papers on blow-up behaviors.

In [5] Galaktionov showed the initial boundary value problem of the equation $u_t = \Delta u^m + u^p$ in convex bounded domains has global nontrivial solution if $p < m$; and the blow-up may occur if $p \geq m$. In this article we consider the position of blow-up points, the asymptotic behavior of the solution near the blow-up time T: furthermore, we disclose an isoperimetric problem for the above equation.

Throughout we suppose Ω is a convex bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$ and $p > m$ such that the blow-up appears.

2. The Position of Blow-up Points

Consider the initial boundary value problem

$$\begin{cases} u_t = \Delta u^m + u^p & (x,t) \in \Omega \times (0,T) \equiv Q_T \\ u(x,t) = 1 & x \in \partial\Omega \\ u(x,0) = \varphi(x) \geq 1 \end{cases} \tag{2.1}$$

We consider positive solutions of the problem. Then the solution can not take positive minimum in Q_T. So we have $u > 1$ in Q_T.

Assume $\Delta\varphi^m + \varphi^p \geq 0$. By the same consideration we have $u_t > 0$ in Q_T. We shall show blow-up points can not belong to ∂Q_T.

Take any point $y_0 \in \partial\Omega$. Without loss of generality we can choose $y_0 = 0$ after fixing it. Since Ω is convex, we may assume $\partial\Omega$ is tangent to the hyperplane $x_1 = 0$ at $y_0 = 0$ and lies below this plane. Denote

$$\Omega_\alpha^+ = \{x : x \in \Omega,\ x_1 > \alpha\},\ \alpha < 0;\ \Omega_\alpha^- = \{x : (2\alpha - x_1, x') \in \Omega_\alpha^+\}, x' = (x_2, \cdots, x_n)$$

Note $2\alpha - x_1 < \alpha < x_1$. Introduce an auxilliary function

$$w(x,t) = u(x_1, x', t) - u(2\alpha - x_1, x', t)$$

For $x \in \Omega_\alpha^-$ we have

$$w_t - \Delta f(\xi) w = g(\eta) w, \quad f(\xi) = m\xi^{m-1}, g(\eta) = p\eta^{p-1}$$

where $\xi, \eta \in [u(2\alpha - x_1, x', t), u(x_1, x', t)]$, $f > 0, g > 0$.

Since $w = 0$ on the plane $x = \alpha$ and $w = u(x_1, x', t) > 0$ on the boundary $(\partial\Omega_\alpha^- \cap \{x_1 < \alpha\}) \times (0,T)$, hence $w > 0$ in $\Omega_\alpha^- \times (0,T)$; so that actually $u(2\alpha - x_1, x', t) < u(x_1, x', t)$. This means $\dfrac{\partial w}{\partial x_1} = 2\dfrac{\partial u}{\partial x_1} < 0$ on the plane $x_1 = \alpha$.

Because α is arbitrary, there is a $\alpha_0 < 0$, such that $\dfrac{\partial u}{\partial x_1} < 0$, $x \in \Omega_{\alpha_0}^+$, $0 < t < T$.

Recall x_1 is the outer normal at $y_0 = 0$. So we have following result.

Lemma 2.1 *Suppose* $\Delta\varphi^m + \varphi^p \geq 0$. *Then for positive solutions of the problem* (2.1) *and for each point* $P_0 = (x,t) \in \partial\Omega \times (0,T)$, *there exists a* $\alpha_0 < 0$. *Such that* $\dfrac{\partial u}{\partial n_{P_0}} < 0$, $(x,t) \in \Omega_{\alpha_0}^+ \times (0,T)$.

Now we can state and prove the main result of this section.

Theorem 2.2 *The blow-up points of the solution for the problem* (2.1) *are contained in a compact subset of the domain* Ω.

Proof Denote $\alpha = \min\limits_{y_0 \in \partial\Omega} |\alpha_0|$. It is obvious $\alpha > 0$. Introduce the auxilliary function

$$J_1 = v_{x_1} + c(x_1 + \alpha) v^\eta, \quad v = u^m$$

where c and η are positive constants to be determined.

By calculation we get

$$J_t - mv^{\frac{m-1}{m}} \Delta J \leq b_1 J + (m\eta - p) v^{\frac{p-1}{m}} + 2\eta m c v^{\frac{m\eta-1}{m}}, \quad |b_1| \leq k$$

Choose c sufficiently small and $\eta > 1$ such that $m\eta < p$. The above inequality becomes

$$J_t - mv^{\frac{m-1}{m}} \Delta J - b_1 J \leq 0, \quad (x,t) \in \Omega_{1-\alpha}^+ \times (0,T)$$

Since $v < 0$ and $J < 0$ on $\{x_1 = -\alpha\}$ and on $t = 0$ (recall $v_{x_1} = mu^{m-1} u_{x_1} < 0$), so by the maximum principle we get

$$J < 0 \quad \text{in } \Omega_\alpha^+ \times (0,T)$$