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FIRST ORDER SYSTEM OF EQUATIONS OF MIXED TYPE AND SECOND ORDER EQUATION OF MIXED TYPE

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Abstract A system of first order equations of mixed type, which may be reduced to a general second order equation of mixed type, is considered. Uniqueness of solution to the generalized Tricomi problem is proved by the method of auxiliary function. Existence of H^1 strong solution is based on a characteristic problem and is proved by the Fredholm's alternative properties.

Key Words System of mixed type; generalized Tricomi problem; auxiliary function; Fredholm's alternative properties.

Classification 35M05

1. Introduction

In domain $\mathcal{D} \subset R^2$ consider a system of first order equations

$$\begin{cases} Cu_x - Bu_y - v_y + \alpha u + \beta v = f \\ Au_y - Bu_x + v_x + \gamma u + \delta v = g \end{cases} \tag{1}$$

where the coefficient determinant $\Delta \equiv B^2 - AC$ is just the discriminant of type of the system of equations, the system (1) is elliptic when $\Delta < 0$, and is hyperbolic when $\Delta > 0$. If $\Delta < 0$ in a subdomain $\mathcal{D}_+$ of $\mathcal{D}$ and $\Delta > 0$ in another subdomain $\mathcal{D}_-(\equiv \mathcal{D}\backslash\mathcal{D}_+)$ of $\mathcal{D}$, then the system (1) in $\mathcal{D}$ is of mixed type. In hyperbolic domain $\mathcal{D}_-$ two families of characteristics Γ_+ and Γ_- are defined by equations $Adx-(B-\sqrt{\Delta})dy=0$ and $Adx - (B + \sqrt{\Delta})dy = 0$ respectively. For this system of equations of mixed type how pose the problem, this is the problem considered here.

Suppose that functions $A, B, C, \alpha, \beta, \gamma, \delta \in C^1$, $f, g \in H^1$. For (1) we consider a kind of special case:

$$\beta_x + \delta_y = 0 \quad \text{in} \quad \mathcal{D} \tag{2}$$

Now we take a function $\rho(x, y) \in C^1(\mathcal{D})$, which is vanished on the boundary $\partial\mathcal{D}$. Multiplying the first and second equations in (1) by ρ_x and ρ_y respectively, integrating the results over $\mathcal{D}$ and then making integrations of parts (in weak sense) respectively, we have

$$\iint \rho[Cu_{xx} - Bu_{xy} - v_{xy} + (\alpha + C_x)u_x - B_x u_y + \beta v_x + \alpha_x u + \beta_x v - f_x]dxdy = 0$$

$$\iint \rho[Au_{yy} - Bu_{xy} + v_{xy} + (\gamma + A_y)u_y - B_y u_x + \delta v_y + \gamma_y u + \delta_y v - g_y]dxdy = 0$$

Adding together and in consideration of the assumption (2) we get

$$\iint \rho\{Cu_{xx} - 2Bu_{xy} + Au_{yy} + (\alpha + C_x - B_y + \beta B + \delta C)u_x$$
$$+(\gamma + A_y - B_x - \beta A - \delta B)u_y$$
$$+(\alpha_x + \gamma_y - \beta\gamma + \alpha\delta)u - (f_x + g_y - \beta g + \delta f)\}dxdy = 0 \tag{3}$$

In consideration of the arbitrariness of function ρ, we obtain a general second order equation with respect to u:

$$Cu_{xx} - 2Bu_{xy} + Au_{yy} + Eu_x + Du_y + Fu = G \tag{4}$$

where

$$\begin{cases} E \equiv \alpha + C_x - B_y + \beta B + \delta C, & D \equiv \gamma + A_y - B_x - \beta A - \delta B \\ F \equiv \alpha_x + \gamma_y - \beta\gamma + \alpha\delta, & G \equiv f_x + g_y - \beta g + \delta f \end{cases} \tag{5}$$

Since the discriminant $\Delta \equiv B^2 - AC < 0$ in $\mathcal{D}_+$ and > 0 in $\mathcal{D}_-$, hence the equation (4) in $\mathcal{D}(= \mathcal{D}_+ \cup \mathcal{D}_-)$ is a equation of mixed type. For this equation we know that we may consider the generalized Tricomi problem

$$u = 0 \quad \text{on } \Gamma_0 \cup \Gamma'_+ \tag{6}$$

or Tricomi problem

$$u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_+ \tag{7}$$

where Γ_0 is the outer boundary of $\mathcal{D}_+$, which is connected with the degenerating curve $\Delta = 0$ at points A and B; Γ_+ and Γ_- are two families of characteristic curves, issuing from the corresponding points A and B respectively; Γ'_+ is an arbitrary piecewise smooth curve, issuing from the point A and lying inside the characteristic triangle, and defined by the following equation:

$$\Gamma'_+ : \quad dx + p(x,y)dy = 0, \quad p \geq (-B + \sqrt{\Delta})/A \text{ on } \Gamma'_+ \tag{8}$$

2. Uniqueness of Solution

Assume that the coefficients in (4) satisfy the conditions

$$H_1 : \quad D/A = \varphi(y), \quad F/A = \psi(y) \tag{9}$$

Then, we may make transformation

$$u = we^{-\int \mu(y)dy} \tag{10}$$