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THE DIRICHLET PROBLEMS FOR A CLASS OF FULLY NONLINEAR ELLIPTIC EQUATIONS RELATIVE TO THE EIGENVALUES OF THE HESSIAN [1)]

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Abstract In this paper we discuss the Dirichlet problems for a class of fully nonlinear elliptic equations

$$F(D^2u) = \psi(x,u)(\psi(x,u,\nabla u)) \quad \text{in } \Omega$$
$$u = \varphi(x) \quad \text{on } \partial\Omega$$

where F is represented by a symmetric function $f(\lambda_1, \cdots, \lambda_n)$ of the eigenvalues (λ_1, $\cdots$, λ_n) of the Hessian D^2u. This result extends the works of Caffarelli L., Nirenberg L., Spruck L. [2] to more general cases.
Key Words The eigenvalues of the Hessian matrix; admissible; *a priori* estimates for the C^2 norm.
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1. The Main Results

In this paper we extend the results of [2] to more general cases. We will discuss the Dirichlet problem in a bounded domain Ω in R^n with smooth boundary $\partial\Omega$:

$$\begin{aligned} F(D^2u) &= \psi(x,u) \quad \text{in } \Omega \\ u &= \varphi(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

and the Dirichlet problem in a bounded domain Ω in R^n with smooth strictly convex boundary $\partial\Omega$:

$$\begin{aligned} F(D^2u) &= \psi(x,u,\nabla u) \quad \text{in } \Omega \\ u &= \varphi(x) \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

where $\varphi \in C^\infty(\partial\Omega)$ and the function F is of a very special nature. It is represented by a smooth symmetric function $f(\lambda_1, \cdots, \lambda_n)$ of the eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_n)$ of the

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Hessian matrix $D^2u = \{u_{ij}\}$, which we denote by $\lambda(u_{ij})$. The function f is assumed to satisfy

$$f_i = f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} > 0 \text{ for all } i \tag{3}$$

furthermore,

$$f \text{ is a concave function} \tag{4}$$

defined in an open convex cone $\Gamma \overset{\subset}{\neq} R^n$ with vertex at the origin and containing the positive cone $\Gamma^+ : \{\lambda \in R^n |$ each component $\lambda_i > 0\}$. Γ is also supposed to be symmetric in the λ_i.

We assume that for every $C > 0$ and every compact set K in Γ there is a number $R = R(C, K)$ such that

$$f(\lambda_1, \cdots, \lambda_n + R) \geq C \text{ for all } \lambda \in K \tag{5}$$

$$f(R\lambda) \geq C \qquad \text{for all } \lambda \in K \tag{6}$$

Definition 1 *A function $u \in C^2(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$ is called admissible if at every $x \in \overline{\Omega}$, $\lambda(u_{ij})(x) \in \Gamma$.*

Definition 2 *Γ is said to be Type 1 if the positive λ_i axes belong to $\partial\Gamma$; otherwise it is said to be Type 2.*

For Equation (1), we assume that

$$\psi(x, z) \in C^\infty(\overline{\Omega} \times R), \ \psi(x, z) > 0$$
$$\text{and } \psi_z \geq 0 \tag{7}$$

There exists an admissible subsolution $\underline{u} \in C^5(\overline{\Omega})$ such that

$$F(D^2\underline{u}) \geq \psi(x, \underline{u}) \text{ in } \Omega$$
$$\underline{u} = \varphi(x) \qquad \text{on } \partial\Omega \tag{8}$$

Set

$$\min_{\overline{\Omega}} \psi(x, \underline{u}(x)) = \psi_0 > 0$$

We assume that for some $\overline{\psi}_0 < \psi_0$,

$$\overline{\lim_{\lambda \to \lambda_0}} f(\lambda) \leq \overline{\psi}_0 \text{ for every } \lambda_0 \in \partial\Gamma \tag{9}$$

Furthermore we have the following condition: for some constant $C_0 > 0$

$$\sum f_{\lambda_i}\lambda_i \geq C_0 \text{ whenever } f(\lambda) \geq \frac{\overline{\psi}_0 + \psi_0}{2} \tag{10}$$