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NUMERICAL ANALYSIS OF THE FRACTIONAL SEVENTH-ORDER KDV EQUATION USING AN IMPLICIT FULLY DISCRETE LOCAL DISCONTINUOUS GALERKIN METHOD

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Abstract. In this paper an implicit fully discrete local discontinuous Galerkin (LDG) finite element method is applied to solve the time-fractional seventh-order Korteweg-de Vries (sKdV) equation, which is introduced by replacing the integer-order time derivatives with fractional derivatives. We prove that our scheme is unconditional stable and L^2 error estimate for the linear case with the convergence rate $O(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})$ through analysis. Extensive numerical results are provided to demonstrate the performance of the present method.

Key words. Time-fractional partial differential equations; Seventh-order KdV equation; Local discontinuous Galerkin method; Stability; Error estimates.

1. Introduction

Several researchers in fractional calculus mentioned that derivatives of noninteger order are very effective for the description of many physical phenomena such as damping laws, and diffusion process [18, 25]. Some fractional partial differential equations have been solved, such as time-fractional telegraph equation [1], fractional Fokker-Planck equation [5], space-time fractional Schrödinger equation [8, 26], fractional order two point boundary value problem [7], the fractional KdV equation [16], fractional diffusion equation [17, 23], fractional derivative fluid model [9], fractional KdV-Burgers-Kuramoto equation [21] and so on. Machado et al. [14] introduced the recent history of fractional calculus, as for the detailed theory and applications of fractional integrals and derivatives, we can refer to [11, 15, 20] and the references therein. Solving such fractional partial differential by the robust and accurate numerical methods has become popular with their frequent appearance in applied science and engineering.

The KdV type of equations, which were first derived by Korteweg and de Vries (1895) and used to describe weakly nonlinear shallow water waves, have emerged as an important class of nonlinear evolution equation and are often used in pratical applications. The seventh-order KdV (sKdV) equation was first introduced by Pomeau et. al [19] in order to discuss the structural stability of the KdV equation under singular perturbation. Some methods [6, 13] have been used to handle the integer-order equations, however, to the best of our knowledge, the study of the fractional sKdv equations has not been widespread. In this paper, we consider the following generalized time-fractional sKdv equation

$$
\begin{aligned}
&D_t^{\alpha} u(x,t) + g(u)_x + u_{3x} - u_{5x} + \lambda u_{7x} = 0, \\
&u(x,0) = u_0(x),
\end{aligned}
\tag{1.1}
$$

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where λ is anonzero constant. $0 < \alpha \leq 1$ is a parameter describing the order of the fractional time. We do not pay attention to boundary condition in this paper; hence the solution is considered to be either periodic or compactly supported.

The time fractional derivative in the equation (1.1), uses the Caputo fractional partial derivative of order α, defined as [18]

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha} & if\ 0 < \alpha < 1, \\ \frac{\partial u(x,t)}{\partial t} & if\ \alpha = 1, \end{cases} \tag{1.2}$$

here $\Gamma(\cdot)$ is the Gamma function.

The discontinuous Galerkin finite element method is a very attractive method for partial differential equations because it is naturally formulated for any order of accuracy in each element, flexible and efficient in terms of mesh and shape functions. The purpose of the present paper is to solve and analyze time-fractional sKdV equation by introducing an implicit fully discrete local discontinuous Galerkin method. This development is based on the extensive work on DG for problems founded in classic calculus [10, 22, 24, 27]. We prove that our scheme is unconditionally stable and give an error estimate for the linear case.

The remains of this paper are organized as follows. In the next section, we introduce some basic notations and mathematical preliminaries. Then, in Section 3, we discuss the LDG scheme for the fractional equation (1.1), and prove that the scheme is unconditionally stable, and the numerical solution is convergent. Numerical experiments to illustrate the accuracy and capability of the method are given in Section 4. Finally, in Section 5, concluding remarks are provided.

2. Notations and auxiliary results

2.1. Notations. First, the domain Ω is partitioned into elements $\Omega = \bigcup_j I_j$ with a spatial grid $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$. $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \cdots N$. The cell lengths $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \leq j \leq N$, and $h = \max_{1\leq j\leq N} \Delta x_j$. The solution of the numerical scheme is denoted by u_h^n which belongs to the finite element space V_h^k:

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \cdots N\},$$

$P^k(I_j)$ denotes the set of all polynomials of degree at most k on I_j.

For a function $u_h^n \in V_h^k$, We denote the limits at the points $\{x_{j+\frac{1}{2}}\}$ by

$$(u_h^n)^{\pm}_{j+\frac{1}{2}} = \lim_{x \to x^{\pm}_{j+\frac{1}{2}}} u_h^n,$$

$(u_h^n)^-_{j+\frac{1}{2}}$ and $(u_h^n)^+_{j+\frac{1}{2}}$ refer to the value of u_h^n at $x_{j+\frac{1}{2}}$ from the left cell I_j and the right cell I_{j+1}, respectively. The jump $(u_h^n)^+_{j+\frac{1}{2}} - (u_h^n)^-_{j+\frac{1}{2}}$ by $[u_h^n]_{j+\frac{1}{2}}$. The jump will be zero for a continuous function.

2.2. Numerical flux. Consider a scalar conservation law given in differential form

$$\phi_t + g(\phi)_x = 0, \tag{2.1}$$

where $g(\phi)$ is called the flux function. Numerically, $g(\phi)$ should be expressed by a suitable choice at the interface. For discontinuous Galerkin spatial discretization, $g(\phi)$ is approximated by the numerical form at the discontinuous point $x_{j+\frac{1}{2}}$. In this paper, the flux $\widehat{g}(\phi^-, \phi^+)$ will be used to denote the numerical flux, which is