INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 481–507

ON FULLY DISCRETE FINITE ELEMENT SCHEMES FOR EQUATIONS OF MOTION OF KELVIN-VOIGT FLUIDS

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Abstract. In this paper, we study two fully discrete schemes for the equations of motion arising in the Kelvin-Voigt model of viscoelastic fluids. Based on a backward Euler method in time and a finite element method in spatial direction, optimal error estimates which exhibit the exponential decay property in time are derived. In the later part of this article, a second order two step backward difference scheme is applied for temporal discretization and again exponential decay in time for the discrete solution is discussed. Finally, a *priori* error estimates are derived and results on numerical experiments conforming theoretical results are established.

Key words. Viscoelastic fluids, Kelvin-Voigt model, a *priori* bounds, backward Euler method, second order backward difference scheme, optimal error estimates.

1. Introduction

In this article, we discuss the convergence of the backward Euler method and the second order backward difference scheme for the following system of equations of motion arising in the Kelvin-Voigt fluids (see [19]):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t), \;\; x \in \Omega, \;\; t > 0, \tag{1.1}$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \;\; \mathbf{x} \in \Omega, \, t > 0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega, \; t \geq 0, \tag{1.3}$$

where, Ω is a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$. Here $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity vector, $p = p(\mathbf{x}, t)$ the pressure and $\nu > 0$, the kinematic coefficient of viscosity. Moreover, the velocity of the fluid, after instantaneous removal of the stress, does not vanish instantaneously but dies out like $exp(\kappa^{-1}t)$ (see [19]), where κ is the retardation parameter. For details of the physical background and its mathematical modeling, we refer to [6]-[7] and [9]. Throughout this paper, we assume that the right hand side function $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term.
Based on the analysis of Ladyzenskaya [17] for the solvability of the Navier Stokes equations, Oskolkov [18, 19], has proved the global existence of a unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1.1)-(1.3). The investigations on solvability are further continued by him and his collaborators, see [21] and [22] and they have discussed the existence and uniqueness results on the entire semiaxis $\mathbb{R}^+$ in time.
For the related literature on the time discretization of equations of motion arising in the viscoelastic model of Oldroyd type see [2], [13], [24] and [26]-[29]. Interestingly, there is hardly any work devoted to the time discretization of (1.1)-(1.3). For the earlier results on the numerical approximations to the solutions of the problem

Received by the editors April 19, 2012.
2000 *Mathematics Subject Classification.* 65M60,65M12,65M15,35D05,35D10.

(1.1)-(1.3), we refer to [3] and [20]. Under the condition that the solution is asymptotically stable as $t \to \infty$, the authors of [20] have established the convergence of spectral Galerkin approximations for the semi axis $t \geq 0$. Recently, Bajpai et al. [3] have applied finite element methods to discretize the spatial variables and derived optimal error bounds for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$- norm. In [3] and [20], only semidiscrete approximations for (1.1)-(1.3) are discussed, keeping the time variable continuous. In this article, we have discussed both backward Euler method and two step backward difference scheme for the time discretization and have derived optimal error estimates. We have also discussed briefly, the proof of linearized backward Euler method applied to (1.1)-(1.3) for time discretization. More precisely, we have

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the backward Euler or linearized backward Euler method.

In the later part of this article, we have proved the following result for a second order backward difference scheme:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k^2) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k^{2-\gamma}),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the second order backward difference scheme and

$$\gamma = \begin{cases} 0 & \text{if } n \geq 2; \\ 1 & \text{if } n = 1. \end{cases}$$

The remaining part of this paper is organized as follows. In Section **2**, we discuss the preliminaries. In Section **3**, we derive *a priori* bounds for the semidiscrete solutions and present some spatial error estimates required for error analysis. In Section **4**, we obtain a *priori* bounds for the discrete solution and prove the existence and uniqueness of the discrete solution. In Section **5**, we establish the error estimates for the velocity and pressure of the backward Euler method. Section **6** deals with the error estimates for velocity and pressure using the second order backward difference scheme. In Section **7**, we provide some numerical results to confirm our theoretical results.

2. Preliminaries

For the mathematical formulation of (1.1)-(1.3), we denote $\mathbb{R}^d,\ (d = 2, 3)$-valued function spaces using boldface letters. That is,

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \text{ and } \quad \mathbf{H}^m = (H^m(\Omega))^d,$$

where $L^2(\Omega)$ is the space of square integrable functions defined in Ω. The space $L^2(\Omega)$ is a Hilbert space endowed with the usual scalar product $(\phi, \psi) = \displaystyle\int_\Omega \phi(x)\psi(x)\, dx$ and the associated norm $\|\phi\| = \left(\displaystyle\int_\Omega |\phi(x)|^2\, dx\right)^{1/2}$. Further, $H^m(\Omega)$ is the standard Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $\|\phi\|_m =$