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THE FIRST BOUNDARY VALUE PROBLEM FOR GENERAL PARABOLIC MONGE-AMPERE EQUATION*

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Abstract In this note we consider the first boundary value problem for a general parabolic Monge-Ampere equation

$$u_t - \log\det(D_{ij}u) = f(x,t,u,D_x u) \text{ in } Q, \quad u = \varphi(x,t) \text{ on } \partial_p Q$$

It is proved that there exists a unique convex in x solution to the problem from $C^{4+\beta,2+\beta/2}(\overline{Q})$ under certain structure and smoothness conditions (H3)−(H7).

Key Words General parabolic Monge-Ampere equation; first boundary value problem; classical solution.

Classifications 35K20; 35K55; 35Q99.

1. Introduction

In [1] it is proved that the first boundary value problem for the parabolic Monge-Ampere equation

$$\begin{cases} -D_t u \det(D_{ij}u) = f(x,t) & \text{in } Q \\ u = \varphi(x,t) & \text{on } \partial_p Q \end{cases} \tag{1.1}$$

has a unique solution in $C^{4+\alpha,2+\alpha/2}(\overline{Q})$. Here $Q=\Omega\times(0,T]$, Ω is a bounded convex domain in R^n, $\partial_p Q=(\partial\Omega\times[0,T])\cup(\Omega\times\{t=0\})$ is the parabolic boundary of Q, $T>0$, $\alpha\in(0,1)$ are constants.

The aim of this note is to extend the result in [1] to the parabolic Monge-Ampere equation of more general form. We mainly discuss the first boundary value problem for the parabolic Monge -Ampere equation of the form

$$\begin{cases} D_t u - \log\det(D_{ij}u) = f(x,t,u,D_x u) & \text{in } Q \\ u = \varphi(x,t) & \text{on } \partial_p Q \end{cases} \tag{1.2}$$

Hereafter, besides using the notations, terminologies and conventions in [1] [2] with-

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out indication, we denote the norm in $C^{k,k/2}(\overline{Q})$ by $|\cdot|_{k,k/2}$ for $k\in R$, and $D_x=(D_1,\cdots,D_n)$, $D_i=\frac{\partial}{\partial x_i}$, for $i=1,\cdots,n$.

Although the corresponding extension for elliptic Monge-Ampere equation has been completed in [2] by Caffarelli, Nirenberg and Spruck, and though the idea in [2] also can be followed, there are still some differences in dealing with the parabolic counterpart. Firstly, since the "compatibility condition" should be fulfilled for the first boundary value problem in a cylindrical domain for an equation of parabolic type, the structure conditions for (1.2) given in this note are different from those for the elliptic case in [2]. Secondly, since the outward normal to the boundary of Q can not be defined at the lateral edge of the lower base of the cylinder Q, we can not use the same definition as in [2] to construct the open set $\mathscr{S}$. What we have to do here is to add more restrictions on the element of $\mathscr{S}$ which guarantee that $\mathscr{S}$ is an open set in $C_0^{4,2}(\overline{Q})$ (cf. Lemma 2.7 below).

It is interesting that the structure conditions for (1.2) given in this note play an essential role in constructing $\mathscr{S}$ as well as in defining the topological degree.

In order to follow the idea in [2] to employ the topological degree theory a mapping must be considered. At that moment we need an existence and uniqueness theorem for the problem

$$\begin{cases} D_t u - \log\det(D_{ij}u) = f(x,t), & \text{in } Q \\ u=\varphi(x,t) & \text{on } \partial_p Q \end{cases} \tag{1.3}$$

under the hypotheses:

(H1) $F(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$.

(H2) φ and f satisfy the compatibility conditions

$$\begin{cases} D_t\varphi - \log\det(D_{ij}\varphi) = f(x,t) \\ D_tD_t\varphi - \varphi^{ij}D_{ij}(f+\log\det(D_{ij}\varphi)) = D_tf(x,t) \end{cases}$$
$$\text{for } (x,t)\in\partial\Omega\times\{t=0\}$$

where $(\varphi^{ij})=(D_{ij}\varphi)^{-1}$.

(H3) $Q=\Omega\times(0,T]$, where $T>0$ is a constant, Ω is a uniformly convex $C^{4+\alpha}$ domain in Euclidian n space R^n, i.e. there exists a function $r(x)\in C_{loc}^{4+\alpha}(R^n)$ such that $\Omega=\{x\in R^n; r(x)<0\}$ and that

$$(D_{ij}r(x))\geq\mu I,\ |D_xr|\geq\mu,\ \text{for } x\in\partial\Omega$$

where $\alpha\in(0,1)$, $\mu>0$ are constants, I is the $n\times n$ unit matrix.

(H4) $\varphi(x,t)\in C^{4+\alpha,2+\alpha/2}(\overline{Q})$, $(D_{ij}\varphi)\geq\mu I$, for $(x,t)\in\overline{Q}$.

The theorem can be proved in the same way as in [1] via the results from [2]—[8], so we ommit its proof and only formulate its statement here.

Theorem 1.1 *If (H1)—(H4) hold, $\alpha\in(0,1)$, then the problem (1.3) has a unique*