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NONLINEAR DEGENERATE OBLIQUE BOUNDARY VALUE PROBLEMS FOR SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS*

Bao Jiguang
(Beijing Normal University)

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Abstract In this paper we study the existence theorem for solution of the nonlinear degenerate oblique boundary value problems for second order fully nonlinear elliptic equations

$$F(x,u,Du,D^2u)=0 \quad x\in\Omega,$$
$$G(x,u,D_\tau u)=0, \quad x\in\partial\Omega$$

where $F(x,z,p,r)$ satisfies the natural structure conditions, $G(x,z,q)$ satisfies $G_q\geqslant 0, G_z\leqslant -G_0<0$ and some structure conditions, vector τ is nowhere tangential to $\partial\Omega$. This result extends the works of Lieberman G. M., Trudinger N. S. [2], Zhu Rujin [1] and Wang Feng [6].

Key Words Nonlinear degenerate oblique value problems; oblique derivative boundary estimate.

Classification 35J.

In papers [1] and [2] the authors proved respectively the existence theorems for solution of the general boundary value problems for second order quasilinear elliptic equations and the nonlinear oblique boundary value problems for second order fully nonlinear elliptic equations. Here we are concerned with the nonlinear degenerate oblique boundary value problems for second order fully nonlinear elliptic equations of the form

$$F(x,u,Du,D^2u)=0, \quad x\in\Omega \tag{1}$$
$$G(x,u,D_\tau u)=0, \quad x\in\partial\Omega \tag{2}$$

where Ω is a bounded domain in R^n with boundary $\partial\Omega\in C^4$, F and G are respectively twice, threetimes continuously differentiable functions on the domains $\Gamma=\Omega\times R^1\times R^n\times \$^n$, $\Gamma'=\partial\Omega\times R^1\times R^1$. Here $\$^n$ is the $n(n+1)/2$ dimensional linear space of $n\times n$ real symmetric matrices, $\tau\in C^2(\overline{\Omega})$ is a unit vector, $Du=(D_iu)$ and $D^2u=[D_{ij}u]$ are gradient vector and Hessian matrix of the function u, $D_\tau u=\tau\cdot Du$.

We denote by (x,z,p,r), (x,z,q) the points in Γ,Γ'. The operator F is elliptic at

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(x,z,p,r) if the matrix $F_r=[F_{r_{ij}}]$ is positive at (x,z,p,r). We shall always suppose that F is elliptic at every point in Γ, and let λ, Λ denote the minimum and maximum eigenvalues of F_r. Here we suppose only that $G_q \geqslant 0$ on Γ' which is different from the oblique boundary value problems.

Suppose that F satisfies the natural structure conditions introduced by Lieberman G. M. and Trudinger N. S. in [2]:

$$
\begin{aligned}
&(F1) \quad \Lambda \leqslant \lambda\mu(|z|)\\
&(F2) \quad |F(x,z,p,0)| \leqslant \lambda\mu_0(|z|)(1+|p|^2)\\
&(F3) \quad (1+|p|)|F_p|, |F_z|, |F_x| \leqslant \lambda\mu_1(|z|)(1+|p|^2+|r|)\\
&\qquad\quad F_z(x,z,0,0) \leqslant -F_0\\
&(F4) \quad |F_{rp}|, |F_{rz}, |F_{rx}| \leqslant \lambda\mu_2(|z|+|p|)\\
&\qquad\quad |F_{pp}|. |F_{pz}|, |F_{px}|, |F_{zz}|, |F_{zx}|, |F_{xx}| \leqslant \lambda\mu_2(|z|+|p|)(1+|r|)\\
&(F5) \quad F_{rr} \leqslant 0
\end{aligned}
$$

for all $(x,z,p,r) \in \Gamma$, where μ, μ_0, μ_1, μ_2 are nondecreasing functions and F_0 is a positive constant.

For G we formulate the structure conditions analogous to [1] and [3]:

$$
\begin{aligned}
&(G1) \quad G_q \geqslant 0, \quad \tau \cdot \nu \geqslant \rho\\
&(G2) \quad G_z(x,z,0) \leqslant -G_0\\
&\qquad\quad G_z \leqslant -G_0, \quad \text{for } (z-z_0)q > 0, \quad |q| \geqslant q_0\\
&(G3) \quad |G_q|, |G_{qx}|, |G_{qxx}| \leqslant \mu_3(|z|)\\
&\qquad\quad |G|, |G_x|, |G_z|, |G_{xz}|, |G_{zz}|, |G_{xx}|, |G_{zzz}| \leqslant \mu_3(|z|)(1+|q|)
\end{aligned}
$$

for all $(x,z,q) \in \Gamma'$, where ν is the unit inner normal to $\partial\Omega, \rho, G_0, q_0$ are positive constants, $z_0=z_0(x)$ satisfies $G(x,z_0,0)=0$ on $\partial\Omega$ and μ_3 is a nondecreasing function.

In order to use the known results and obtain the existence theorem for solution of (1), (2), we consider the following approximate problem

$$F(x,u,Du,D^2u)=0, \quad x \in \Omega \tag{3}$$

$$G(x,u,D_\tau u)+D_\tau u/m=0, \quad x \in \partial\Omega \tag{4}$$

for $m=1,2,\cdots$. From the maximum modulus estimate and the existence theorem [2, Lemma 7.1 and Corollary 7.10], the interior estimate [4, Theorem 8.1], the interior regularity theorem [5, Lemma 17.16], the problem (3), (4) has a solution $u_m: u_m \in C^{2,\theta}(\bar{\Omega}) \cap C^{3,\theta}(\Omega)$ for some $\theta \in (0,1)$, and

$$\sup_{\Omega}|u_m| \leqslant M_0 = \max\left\{\frac{1}{F_0}\sup|F(x,0,0,0)|, \frac{1}{G_0}\sup|G(x,0,0)|\right\} \tag{5}$$

$$|u_m|_{C^{2,\theta}(\bar{\Omega}')} \leqslant M(\Omega')$$

for any $\Omega' \subset\subset \Omega$, where θ depends only on n, μ and $M(\Omega')$ depends also on μ_0, μ_1, μ_2,