J. Partial Differential Equations
Vol. 2, No. 3 (1989), 23—29

# LIE'S TRANSFORMATION GROUP AND SIMILARITY SOLUTIONS OF THE $S_3$ EQUATION①

Pan Zuliang (Zhejiang Univ.)
Zheng Kejie (Post-Telecommunication Institute of Zhejiang)

(Received May 31, 1987; revised April 12, 1988)

Lie's transformation group method is an effective approach of finding the symmetry and particular solutions with some physical meaning of the nonlinear differential equations. It has been widely applied to the soliton theory, the nonlinear diffusion equations and the free boundary problems, etc. [1]-[5] In this paper, we have applied Lie's method to the nonlinear $S_3$ equation in a quasi one-dimensional crystal and obtained some significant results in both studying the algebraic geometry property and finding useful particular solutions for the equation.

Let us consider the $S_3$ equation[6]

$$\begin{aligned} i\varphi_t(x,t) &= a\varphi(x,t) - b\varphi_{xx}(x,t) \\ &\quad - c|\varphi(x,t)|^2\varphi(x,t) \end{aligned} \tag{1}$$

which is invariant with respect to the following transformation

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x,t,\varphi) + O(\varepsilon^2) \\ \bar{t} &= t + \varepsilon\tau(x,t,\varphi) + O(\varepsilon^2) \\ \bar{\varphi} &= \varphi + \varepsilon\eta(x,t,\varphi) + O(\varepsilon^2) \end{aligned} \tag{2}$$

so that it is necessary for

$$i[\eta_t] - a\eta + b[\eta_{xx}] + c[2|\varphi|^2\eta + \eta^*\varphi^2] = 0 \tag{3}$$

where $[\eta_t]$ and $[\eta_{xx}]$ are the infinitesimals of $\varphi_t, \varphi_{xx}$ respectively under the transformation (2). $\eta^*$ is the complex conjugate of $\eta$. By means of the well-known method[7], we obtain

$$\begin{cases} \xi = c_1x + 2bc_2t + c_3 \\ \tau = 2c_1t + c_4 \\ \eta = \{-c_1 + i(c_2x - 2ac_1t + c_5)\}\varphi \end{cases} \tag{4}$$

① Presented at the DD7 Symposium, Tianjin, 1986.

This work was supported by National Science Foundations.

where $c_i (i=1,2,3,4,5)$ are arbitrary real constants.

Next, we solve the following characteristic equations

$$\frac{dx}{c_1 x + 2bc_2 t + c_3} = \frac{dt}{2c_1 t + c_4} = \frac{d\varphi}{\{-c_1 + i(c_2 x - 2ac_1 t + c_5)\}\varphi} \tag{5}$$

the following similarity variable and similarity solution form can be obtained

$$\xi = \frac{1}{\sqrt{2c_1 t + c_4}}\left\{x - \frac{2bc_2}{c_1}t - \frac{1}{c_1^2}(2bc_2c_4 - c_1c_3)\right\} \tag{6}$$

$$\varphi = f(\xi)\exp\{(B - Ac_4)\ln(2c_1 t + c_4) + A(2c_1 t + c_4) + i\frac{c_2}{c_1}(x - mt - n)\} \tag{7}$$

where

$$m = 2bc_2/c_1 \;,\; n = (2bc_2c_4 - c_1c_3)/c_1^2$$
$$A = i(c_2 m - 2ac_1)/4c_1^2$$
$$B = -1/2 + i(c_2 n + c_5)/2c_1$$

Substituting Eqs. (6) and (7) into Eq. (1), the function $f(\xi)$ must satisfy the following ordinary differential equation

$$bf''(\xi) - ic_1\xi f'(\xi) + Nf(\xi) + c|f(\xi)|^2 f(\xi) = 0 \tag{8}$$

where

$$N = -ic_1 - \{c_1^2(c_5 + ac_4) + c_2(bc_2c_4 - c_1c_3)\}/c_1^2$$

The generators of the transformation group (2) are

$$P_1 = \frac{\partial}{\partial x}\;,\; P_2 = \frac{\partial}{\partial t}\;,\; P_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - (1 + 2ait)\varphi\frac{\partial}{\partial \varphi}$$

$$P_4 = 2bt\frac{\partial}{\partial x} + ix\frac{\partial}{\partial \varphi}\;,\; P_5 = i\varphi\frac{\partial}{\partial \varphi}$$

It is easily proved that they constitute a closed Lie algebra. Now, we discuss it as follows:

(i) $c_1 \neq 0, \quad c_i = 0 \quad (i=2,3,4,5)$

Similarity variable and similarity solutions are respectively

$$\xi = x/\sqrt{t} \tag{9}$$

$$\varphi = e^{-iat} f(\xi)/\sqrt{t} \tag{10}$$

Then $f(\xi)$ satisfies the following ordinary differential equation

$$bf'' - \frac{1}{2}i\xi f' - \frac{1}{2}if + c|f|^2 f = 0 \tag{11}$$

In general, Eq. (11) does not belong to the Painlevé type[8].

(ii) $c_1 = 0 \quad , c_i \neq 0 \qquad (i=2,3,4,5)$