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INITIAL-BOUNDARY VALUE PROBLEM FOR DOUBLE DEGENERATE NONLINEAR PARABOLIC EQUATION①

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In this paper we consider the following initial-boundary value problem for double degenerate nonlinear parabolic equation

$$\frac{\partial(|u|^{q-2}u)}{\partial t}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+\varphi(u)\ ,\ x\in\Omega,\ 0<t\leq T \qquad (1.1)$$

$$u(x,t)=0\ ,\ x\in\partial\Omega\ ,\ 0<t\leq T \qquad (1.2)$$

$$u(x,0)=u_0(x)\ ,\ x\in\Omega \qquad (1.3)$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$ and p, q, α are nonnegative real numbers,

$$p,q\geq 2\ ;\ q<\min\left(p+1,\ p+\frac{p}{n}\right);\ \varphi(u)=u^{1+\alpha},\ \alpha>0$$

Equation (1.1) will be degenerate when $u=0$ or $\frac{\partial u}{\partial x_i}=0\ (i=1,2,\dots,n)$. Let $v=|u|^{q-2}u$, (1.1) can be transformed to

$$\frac{\partial v}{\partial t}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial}{\partial x_i}(|v|^{q'-2}v)\right|^{p-2}\frac{\partial}{\partial x_i}(|v|^{q'-2}v)\right]+\varphi(|v|^{q'-2}v)\ ,\ \frac{1}{q}+\frac{1}{q'}=1$$

When $p=2$, it is called filtration equation with absorption, the corresponding initial-boundary problem has been studied in [1], [2]. If $q'=q=2$, the corresponding initial-boundary problem has heen studied in [3], [4]. For general case, Raviart [5], [6] used the method of difference and the method of monotone to prove the existence of solution for the problem (1.1)—(1.3) when $\varphi(u)=0$.

In this work we use the method of "partial regularization", the compactness theorem and the method of monotone to study the existence of global solution, local solution and blowing up of the problem (1.1) — (1.3). In the end, the uniqueness theorem for

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one-dimension case is proved.

In the following:

Section 1 is devoted to preliminaries.

Section 2 investigates global existence theorem when $2+a<p$, or $2+a\leq q$.

Section 3 investigates global existence theorem when $\max(p, q)<2+a<\frac{np}{n-p}$.

In Section 4 local existence for the case $\max(p, q)<2+a<p+\frac{pq}{n}$ and blowing up of solution are discussed.

In Section 5 the uniqueness of solution for $\frac{\partial(|u|^{q-2}u)}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$ with condition (1.2), (1.3) is proved.

1. Preliminaries

We shall use the notations in [7]. Let $W_0^{1,p}(\Omega)$ be a Sobolev space, $A(u)=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)$.

Lemma 1 *Operator A: $W_0^{1,p}(\Omega)\to W_0^{1,p}(\Omega)$ is bounded monotone and semicontinuous.* For proof we refer to [7] (Ch. II, Section 1).

The following lemma of compactness will play an important role in limiting process. Let B, B_1 be Banach spaces, $B\subset_{\to} B_1$ is continuous imbedding, S is a subset in B and satisfies $\lambda S\subset S$ (λ is constant), $M(v)$ is a function, defined on S and

$$S\subset B\subset B_1;\ M(v)\geq 0 \text{ on } S;\ M(\lambda v)=|\lambda|M(v) \tag{1.4}$$

$$\{v\,|\,v\in S,\ M(v)\leq 1\} \text{ is compact with respect to } B \tag{1.5}$$

then the following lemma holds (see [7], Ch. I, Section 12).

Lemma 2 *If (1.4) and (1.5) hold, put*

$$\mathscr{F}=\{v\,|\,v \text{ take value in } B_1, \text{ local measurable in } [0,T]$$

$$\int_0^T[M(v(t))]^{s_1}dt\leq c,\ \int_0^T\left\|\frac{\partial v(t)}{\partial t}\right\|_{B_1}^{s_2}dt\leq c\},\ 1<s_1, s_2<\infty$$

then $\mathscr{F}\subset_{\to} L^{s_1}(0, T; B)$ and the imbedding is compact.

Lemma 3 *Put*

$$S=\{v\,|\,|v|^{q-2}v\in W_0^{1,p}(\Omega)\}$$

$$M(v)=\left[\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial}{\partial x_i}(|v|^{q-2}v)\right|^p dx\right]^{\frac{1}{(q-1)p}}$$

If $(q-1)p>1$, then the set $\{v\,|\,v\in S,\ M(v)\leq 1\}$ is compact with respect to $L^{(q-1)p}(\Omega)$.