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INITIAL AND NONLINEAR OBLIQUE BOUNDARY VALUE PROBLEMS FOR FULLY NONLINEAR PARABOLIC EQUATIONS *

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Abstract

We consider the initial and nonlinear oblique derivative boundary value problem for fully nonlinear uniformly parabolic partial differential equations of second order. The parabolic operators satisfy natural structure conditions which have been introduced by Krylov. The nonlinear boundary operators satisfy certain natural structure conditions also.

The existence and uniqueness of classical solution are proved when the initial, boundary values and the coefficients of the equation are suitable smooth.

1. Introduction

In this paper we are concerned with initial and oblique nonlinear parabolic boundary value problems of the general form

$$F(x,\ t,\ u,\ D_xu,\ D_x^2u) - u_t = 0,\quad (x,\ t) \in Q = \Omega \times (0,\ T] \tag{1. 1}$$

$$u|_{t=0} = u_0(x) \qquad ,\ x \in \Omega \tag{1. 2}$$

$$G(x,\ t,\ u,\ D_xu) = 0 \qquad ,\ (x,\ t) \in \partial\Omega \times [0,\ T] \tag{1. 3}$$

where Ω is a bounded smooth domain in Euclidean n space R^n, and F, G are real valued functions on the domains $\Gamma = \bar{Q} \times R \times R^n \times \$^n$ and $\Gamma' = \partial\Omega \times [0,\ T] \times R \times R^n$ respectively. Here $\$^n$ denotes the $n(n+1)/2$ dimensional linear space of $n \times n$ real symmetric matrices and $D_xu = (u_{x_i})$ $(1 \leq i \leq n)$ and $D_x^2u = [u_{x_ix_j}]$ $(1 \leq i,\ j < n)$ denote the gradient and the Hessian matrix of the real valued function u with respect to the x coordinates. $u_0(x)$ is a real function on Ω. By a classical solution of (1. 1), (1. 2), (1. 3) we shall mean a function $u \in C(\bar{Q}) \cap C^1(\bar{\Omega} \times (0,\ T]) \cap C^2(Q)$ which satisfies (1. 1), (1. 2) and (1. 3) in a pointwise sense.

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Let $X=(x, t, z, p, r)$, $X'=(x, t, z, p)$ denote points in Γ, Γ'. We shall adopt the following definitions of parabolicity and obliqueness for functions F, G differentiable with respect to r, p respectively. Namely, the operator F-u_t is parabolic at $X \in \Gamma$ if the matrix $(F_{r_{ij}})$ is positive at X while the operator G is oblique at $x' \in \Gamma'$ if $\chi = G_{p_\gamma} = G_p \cdot \gamma$ is positive at X'. where γ is the unit inner normal to $\partial\Omega$. We shall call F uniformly parabolic with respect to some subset $U \in \Gamma$ if the ratio of the maximum eigenvalue of $(F_{r_{ij}})$ and the minimum eigenvalue of $(F_{r_{ij}})$ is bounded on U. And call the boundary condition uniform if the ratio of the maximum and the minimum values of χ is bounded.

We shall treat here uniformly parabolic operators F and uniformly oblique operators G subject to certain natural structure conditions. For the operators F these conditions were introduced by Krylov [2] as follows.

Let positive constants λ, Λ, μ_1, μ_2, μ_3 be given and $\lambda \le \Lambda$.

Let $F(x, t, u, u_i, u_{ij})$ $(1 \le i, j \le n)$ be of class C^2 with respect to x, u, $p=u_i$, $r=u_{ij}$ and of class C^1 with respect ot t. Assume the following conditions are satisfied for $(x, t) \in Q$, $u, \tilde{u} \in R$, $u_i, \tilde{u}_i, \xi, \tilde{x} \in R^n$, $u_{ij}, \tilde{u}_{ij} \in \$^n$.

$$uF(x, t, u, 0, 0) \le \mu_1 u^2 + \mu_2 \tag{1. 4}$$

$$\lambda|\xi|^2 \le \Sigma F_{r_{ij}}\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1. 5}$$

$$|F(x, t, u, p_j, 0)| \le \Lambda(1+|p|^2) \tag{1. 6}$$

$$|F|+|F_u|+\sum_i|F_{p_i}|(1+|p|)+\sum_i|F_{x_i}|(1+|p|)^{-1}$$
$$\le M_1(u)(1+|p|^2+|r|) \tag{1. 7}$$

$$F_{(\eta)(\eta)} \le M_2(u, p_k)\{\sum_{i,j}|\tilde{u}_{ij}|[\sum_i|\tilde{u}_i|+(1+|r|)(|\tilde{u}|+|\tilde{x}|)]$$
$$+(1+|r|)\sum_i|\tilde{u}_i|^2+(1+|r|^3)(|\tilde{u}|^2+|\tilde{x}|^2)\} \tag{1. 8}$$

$$|F_t| \le M_2(u, p)(1+|r|^3) \tag{1. 9}$$

where $M_1(u)$, $M_2(u, p)$ are bounded functions of their arguments, $\eta=(\tilde{x}, \tilde{u}, \tilde{u}_i, \tilde{u}_{ij})$ and $F_{(\eta)(\eta)}$ denotes the second directional derivative along η.

For oblique boundary operators G we formulate corresponding conditions.

Let $G(x, t, u, u_i)$, which has been extended to a small neighborhood of $\partial\Omega \times [0, T] \times R \times R^n$, be of class C^2 with respect to x, u, u_i and of class C^1 with respect to t and assume the following conditions are satisfied

$$uG(x, t, u, 0) < 0, \quad \text{when} \quad |u| \ge \mu_3 \tag{1. 10}$$

$$\lambda \le G_{p_\gamma}(x, t, u, p) \le \Lambda \tag{1. 11}$$

$$|G|+|G_x|+|G_{uu}|+|G_{px}|+(1+|p|)^{-1}(|G_t|+|G_{x_n}|)+(1+|p|)^2(|G_{xx}|+$$
$$|G_t|)+(1+|p|)(|G_p|+|G_{pu}|)+(1+|p|^2)|G_{pp}| \le M_1(u)(1+|p|) \tag{1. 12}$$