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A Fast Augmented Lagrangian Method for Euler's Elastica Models

Yuping Duan[1,*], Yu Wang[2] and Jooyoung Hahn[3]

[1] *Institute for Infocomm Research, Singapore.*
[2] *Computer Science Department, Technion, Haifa 32000, Israel.*
[3] *Institute for Mathematics and Scientific Computing, University of Graz, Austria.*

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Abstract. In this paper, a fast algorithm for Euler's elastica functional is proposed, in which the Euler's elastica functional is reformulated as a constrained minimization problem. Combining the augmented Lagrangian method and operator splitting techniques, the resulting saddle-point problem is solved by a serial of subproblems. To tackle the nonlinear constraints arising in the model, a novel fixed-point-based approach is proposed so that all the subproblems either is a linear problem or has a closed-form solution. We show the good performance of our approach in terms of speed and reliability using numerous numerical examples on synthetic, real-world and medical images for image denoising, image inpainting and image zooming problems.

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1. Introduction

Suppose that the observed image u_0 is the original image u perturbed by an additive noise η

$$u_0 = u + \eta.$$

The image denoising problems of recovering the image u from the noisy image u_0 are often solved by variational methods and optimization techniques. Among various variational denoising methods, the Rudin-Osher-Fatemi (ROF) method [31] is probably the most successful one, which is defined by minimizing the following functional

$$\min_u \int_\Omega |\nabla u| + \frac{\mu}{2}\int_\Omega (u-u_0)^2, \tag{1.1}$$

*Corresponding author. *Email addresses:* duany@i2r.a-star.edu.sg (Y. Duan), yuwang@cs.technion.ac.il (Y. Wang), JooyoungHahn@gmail.com (J. Hahn)

where μ is a positive parameter and u is defined on a continuous domain $\Omega \subset \mathbb{R}^2$.

The success of the ROF model mainly relies on the total variation (TV) regularization, which enables the ROF model to preserve sharp edges while removing noise. Due to its nice properties, TV-based models have been further extended to vectorial models for color image denoising [6,13] and several fast algorithms are proposed [9–11]. In spite of many advantageous properties, TV-based methods have a common disadvantage: piecewise constant images are favored over piecewise smooth images, which is the so-called staircasing effect. To overcome this drawback, high order models [7,8,12,14,15,24,26,32] are proposed to yield smoother results. As one of them, Euler's elastica model, which is defined based on the curvature of the level curves of images, was first introduced into computer vision by Mumford [28] and successfully applied to a number of applications, such as image restoration [1–3,17], image segmentation [16,27,29] and image inpainting [4,5,14].

Euler's elastica energy can be described by the curvature κ of a smooth curve Γ as the following

$$E(\Gamma) = \int_\Gamma \big(a + b|\kappa|^\beta(s)\big)ds, \tag{1.2}$$

where s is the arc length and a, b are two positive parameters. In the functional (1.2), the first term minimizes the total length and the second term minimizes the power of total curvature. The power β can be set to either $\beta = 1$ as in [26], or $\beta = 2$ as in [14]. In this work, we set $\beta = 2$, but the techniques developed below can be extended to the case $\beta = 1$ without many efforts. The Euler's elastica of all the level curves of an image u can be expressed as

$$E = \int_{l=0}^{L} \int_{\gamma_l : u=l} \big(a + b|\kappa|^\beta(s)\big)dsdl, \tag{1.3}$$

where γ_l is the level curve with $u = l$. Note that the curvature κ can be expressed as a function of u

$$\kappa(u) = \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big). \tag{1.4}$$

Substituting above equation into (1.3) and using the co-area formula yields

$$E(u) = \int_\Omega \Big(a + b\Big|\nabla \cdot \frac{\nabla u}{|\nabla u|}\Big|^\beta\Big)|\nabla u|. \tag{1.5}$$

For image denoising applications, the elastica energy (1.5) can be used as a regularization term. Together with the data fitting term, we can formulate the minimization problem to approximate the noisy image u_0 by Euler's elastica energy as follows

$$\min_u \int_\Omega \Big(a + b\Big(\nabla \cdot \frac{\nabla u}{|\nabla u|}\Big)^2\Big)|\nabla u| + \frac{\mu}{t}\int_\Omega |u - u_0|^t, \tag{1.6}$$

the choice of t is determined by the type of noise in u_0: e.g., $t = 1$ for salt & pepper noise and $t = 2$ for Gaussian white noise.