INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 4, Pages 845–859

UNIFORM CONVERGENCE OF A COUPLED METHOD FOR CONVECTION-DIFFUSION PROBLEMS IN 2-D SHISHKIN MESH

PENG ZHU, ZIQING XIE*, AND SHUZI ZHOU

Abstract. In this paper, we introduce a coupled approach of local discontinuous Galerkin (LDG) and continuous finite element method (CFEM) for solving singularly perturbed convection-diffusion problems. When the coupled continuous-discontinuous linear FEM is used under the Shishkin mesh, a uniform convergence rate $\mathcal{O}(N^{-1}\ln N)$ in an associated norm is established, where N is the number of elements. Numerical experiments complement the theoretical results. Moreover, a uniform convergence rate $\mathcal{O}(N^{-2})$ in L^2 norm, is observed numerically on the Shishkin mesh.

Key words. convection diffusion equation, local discontinuous Galerkin method, finite element method, Shishkin mesh, uniform convergence

1. Introduction

In recent years, the numerical solutions of singularly perturbed boundary value problems have been received much attention and already studied in many papers and books, see for instance [6, 9, 11, 12]. One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior, i.e., the solution varies very rapidly in a very thin layer near the boundary. Traditional methods such as finite element and finite difference methods, do not work well for these problems as they often produce oscillatory solutions which are inaccurate if the perturbed parameter ϵ is small. When ϵ approaches zero, the problem changes from an elliptic equation to a hyperbolic one. Inspired by the great success of the discontinuous Galerkin (DG) method in solving hyperbolic equations, Cockburn and Shu [4], Celiker and Cockburn [3], Xie et al. [13, 14, 15] and Zhang et al. [19] adopted the local discontinuous Galerkin (LDG) method to solve convection-diffusion equations and analyzed the corresponding convergence properties. On the other hand, nonsymmetric discontinuous Galerkin method with interior penalty (the NIPG method), originally designed for elliptic equations, is analyzed by Zarin and Roos [16] for convection-diffusion problems with parabolic layers.

A disadvantage of DG method is that it produces more degrees of freedom than the continuous finite element method (CFEM). With this motivation, our work is to derive a coupled approach of LDG and CFEM and analyze the uniform convergence in a DG-norm under Shishkin mesh for singularly perturbed convection diffusion problems. The basic idea is to decompose the domain into coarse and fine part and the latter is used to simulate the boundary layer. Then the CFEM using linear

Received by the editors November 3, 2011 and, in revised form, December 19, 2012.
2000 *Mathematics Subject Classification.* 65N15, 65N30.

*Corresponding author.

This research was supported by National Natural Science Foundation of China (Grant No.10571046, 10871066, 11171104), Project of Scientific Research Fund of Hunan Provincial Education Department (09K025), Key Laboratory of High Performance Computing and Stochastic Information Processing, Ministry of Education of China, Hunan Normal University, and Scientific Research Fundation of Jiaxing University(Grant No. 70110X05BL,70510017).

elements is adopted in the fine part where the mesh size is comparable with ϵ, and LDG method is used in the coarse part for its stabilization.

A coupled LDG-CFEM approach has also been studied by Perugia and Schötzau [8] for the modeling of elliptic problems arising in electromagnetics. Roos and Zarin [10], Zarin [17] analyzed the NIPG-CFEM coupled method on Shishkin mesh for convection-diffusion problems with exponentially layers or characteristic layers. In this paper, the coupled LDG method is used for the singularly perturbed convection-diffusion equation for the first time to our knowledge. Moreover, distinguished from the general approaches for proving uniform convergence of numerical methods for singularly perturbed problem on layer-adapted meshes, in which solution decomposition is usually necessary, our analysis is based on the uniform error estimates for the interpolation under the Shishkin mesh, which can be reduced by the priori estimate of the solution, i.e.,

$$\left|\frac{\partial^{i+j}u(x,y)}{\partial x^i\partial y^j}(x,y)\right| \leq C\left(1+\epsilon^{-i}e^{-\beta_1(1-x)/\epsilon}\right)\times\left(1+\epsilon^{-j}e^{-\beta_2(1-y)/\epsilon}\right),$$

for i,j satisfying $0\leq i+j\leq 2$. Our method can be generalized to all DG methods belong to the unify framework in [1], including the NIPG method.

The paper is organized as follows. In Section 2, we introduce the coupled LDG and CFEM for the singularly perturbed problems. Then stability and error analysis of the coupled method on Shishkin mesh is given in Section 3. The implementation of our coupled method on Shishkin mesh is presented in Section 4. It aims to validate our theoretical results. Furthermore, the uniform convergence rate $\mathcal{O}(N^{-2})$ in L^2 norm is observed numerically. This phenomenon is not found in [10] and [17].

In the sequel, with C we shall denote a generic positive constant independent of the perturbation parameter ϵ and mesh size.

2. Coupling the LDG and CFEM

Consider the following two-dimensional convection-diffusion problem

$$\begin{cases} -\epsilon\Delta u+\mathbf{b}\cdot\nabla u+cu=f & \text{in } \Omega=(0,1)^2,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $0<\epsilon\ll 1$ is a small positive parameter, $\mathbf{b},c$, and f are sufficiently smooth functions with the following properties

$$\begin{aligned} &\mathbf{b}(x,y)=(b_1(x,y),b_2(x,y))\geq(\beta_1,\beta_2)>(0,0), c(x,y)\geq 0, \forall(x,y)\in\bar{\Omega},\\ &c_0^2(x,y)\equiv(c-\tfrac{1}{2}\nabla\cdot\mathbf{b})(x,y)\geq\gamma_0>0,\quad\forall(x,y)\in\bar{\Omega},\\ &f(0,0)=f(1,0)=f(0,1)=f(1,1)=0, \end{aligned} \tag{2.2}$$

for some constants β_1,β_2 and γ_0. With the assumptions above, it is well-known that there exists a solution u of (2.1) that in general exhibits an exponentially boundary layer near $x=1$ and $y=1$.

The Shishkin Mesh. Define the transition parameter

$$\tau_x=\min\left(\frac{1}{2},\frac{\kappa}{\beta_1}\epsilon\ln N\right),\quad \tau_y=\min\left(\frac{1}{2},\frac{\kappa}{\beta_2}\epsilon\ln N\right),$$

with $\kappa\geq 2$ and divide Ω into four sub-domains

$$\Omega_0=(0,1-\tau_x)\times(0,1-\tau_y),\qquad \Omega_x=(1-\tau_x,1)\times(0,1-\tau_y),$$

$$\Omega_y=(0,1-\tau_x)\times(1-\tau_y,1),\qquad \Omega_{xy}=(1-\tau_x,1)\times(1-\tau_y,1).$$