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Generalized Normal Derivatives and Their Applications in DDMs with Nonmatching Grids and DG Methods

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Abstract. A class of normal-like derivatives for functions with low regularity defined on Lipschitz domains are introduced and studied. It is shown that the new normal-like derivatives, which are called the *generalized normal derivatives*, preserve the major properties of the existing standard normal derivatives. The generalized normal derivatives are then applied to analyze the convergence of domain decomposition methods (DDMs) with nonmatching grids and discontinuous Galerkin (DG) methods for second-order elliptic problems. The approximate solutions generated by these methods still possess the optimal energy-norm error estimates, even if the exact solutions to the underlying elliptic problems admit very low regularities.

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1. Introduction

It is known that normal derivative of function is well defined on boundary of any Lipschitz domain, provided that the underlying function is smooth enough (see, [13, 20, 21]). But, if the function under consideration has low regularity only, an additional assumption is needed to guarantee the existence of this normal derivative. For smooth domains, this topic was studied in details (see Chapter 2 in [20]). However, the results (e.g.,Theorem 7.3) in [20] can not be extended to the case of Lipschitz domain. The main difficulty rests in the fact that normal vector is discontinuous at the corners of nonsmooth boundary. For Lipschitz domain, only a few results on normal derivative of low regularity function have been obtained by [11] and [13].

On the other hand, the normal derivative indeed plays an important role in numerical analysis of boundary value problems. For example, one has to use normal derivatives on the underlying interface in analysis of convergence of DDMs with nonmatching grids and of DG methods for elliptic boundary value problems of second order (see

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[1, 3, 5, 7, 15, 18, 19, 23, 24]). Most existing error estimates for these methods was derived only under the assumption that the underlying analytic solution has high regularity, since, as we mentioned above, the normal derivative is not well defined without such high regularity. But, this assumption can not be satisfied in some applications. This problem was investigated for DDMs with nonmatching grids in [4], in which an error estimate was derived for a particular case with low regularity by using Hilbert interpolation technique. However, Hilbert interpolation technique is not available for the general case that the analytic solution has different regularities on different subdomains.

In the present paper, we try to extend Theorems 6.5 and 7.3 of Chapter 2 in [20] to second-order elliptic problems on Lipschitz domains in a new manner. To this end, we introduce a class of *generalized normal derivative*, which is defined by a Green-like formula. The generalized normal derivative is well defined under very weak assumptions. Some similar functionals with this generalized normal derivative were mentioned in literatures. However, to our knowledge, this kind of functional has never been studied in details before. It will be shown that the generalized normal derivative preserves the main properties of the usual normal derivative, although it can not be understood as the usual normal derivative. Such generalized normal derivative will be used to derive the optimal energy error estimates for the approximations generated by DDMs with nonmatching grids or by the DG methods for second-order elliptic problems with low regularity solution. An advantage of the new approach is that one can work the case that the loading function of the equation does not belong to L^2 space (compare [4]).

The outline of the remainder of the paper is as follows. In Section 2, we introduce generalized normal derivatives and investigate main properties of the generalized normal derivatives. In Section 3, we apply the generalized normal derivative to analyzing convergence of DDMs with nonmatching grids and DG methods for second-order elliptic problems with low regularity solution. A Hilbert interpolation result on subspace is derived in Appendix.

2. Generalized normal derivatives

This section is devoted to introducing and studying a class of generalized normal derivative.

2.1. Sobolev spaces

In the rest of the paper, we will use various Sobolev spaces repeatedly.

Let $\hat{\Omega} \subset \mathscr{R}^n$ $(n = 2,\ 3)$ be a bounded and connected Lipschitz domain with piecewise smooth boundaries. For convenience, a smooth piece of $\partial\hat{\Omega}$ is called a *face* of $\partial\hat{\Omega}$ in the following. In applications, the domain $\hat{\Omega}$ usually represents a convex subdomain of the underlying Lipschitz domain Ω, which includes polygon (in $\mathscr{R}^2$) and polyhedron (in $\mathscr{R}^3$) with planed or curved faces. Denote by $H^\sigma(\hat{\Omega})$ $(\sigma \in [0,2])$ and $H_0^\sigma(\hat{\Omega})$ $(\sigma \in (0,1])$ the usual Sobolev spaces associated with *weak* derivatives (see [2,13,21]). The norm in $H^\sigma(\hat{\Omega})$ is denoted by $\|\cdot\|_{\sigma,\hat{\Omega}}$. Let $L_{loc}(\hat{\Omega})$ denote the space of locally integrable functions on $\hat{\Omega}$,