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A New Family of High Order Unstructured MOOD and ADER Finite Volume Schemes for Multidimensional Systems of Hyperbolic Conservation Laws

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Abstract. In this paper, we investigate the coupling of the Multi-dimensional Optimal Order Detection (MOOD) method and the Arbitrary high order DERivatives (ADER) approach in order to design a new high order accurate, robust and computationally efficient Finite Volume (FV) scheme dedicated to solve nonlinear systems of hyperbolic conservation laws on unstructured triangular and tetrahedral meshes in two and three space dimensions, respectively. The Multi-dimensional Optimal Order Detection (MOOD) method for 2D and 3D geometries has been introduced in a recent series of papers for mixed unstructured meshes. It is an arbitrary high-order accurate Finite Volume scheme in space, using polynomial reconstructions with *a posteriori* detection and polynomial degree decrementing processes to deal with shock waves and other discontinuities. In the following work, the time discretization is performed with an elegant and efficient one-step ADER procedure. Doing so, we retain the good properties of the MOOD scheme, that is to say the optimal high-order of accuracy is reached on smooth solutions, while spurious oscillations near singularities are prevented. The ADER technique permits not only to reduce the cost of the overall scheme as shown on a set of numerical tests in 2D and 3D, but it also increases the stability of the overall scheme. A systematic comparison between classical unstructured ADER-WENO schemes and the new ADER-MOOD approach has been carried out for high-order schemes in space and time in terms of cost, robustness, accuracy and efficiency. The main finding of this paper is that the combination of ADER with MOOD generally outperforms the one of ADER and WENO either because at given accuracy MOOD is

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less expensive (memory and/or CPU time), or because it is more accurate for a given grid resolution. A large suite of classical numerical test problems has been solved on unstructured meshes for three challenging multi-dimensional systems of conservation laws: the Euler equations of compressible gas dynamics, the classical equations of ideal magneto-Hydrodynamics (MHD) and finally the relativistic MHD equations (RMHD), which constitutes a particularly challenging nonlinear system of hyperbolic partial differential equation. All tests are run on genuinely unstructured grids composed of simplex elements.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07, 65Y05, 65Y20, 65Z05

Key words: Finite Volume, high-order, conservation law, polynomial reconstruction, ADER, MOOD, hyperbolic PDE, unstructured meshes, finite volume, one-step time discretization, local continuous space-time Galerkin method, WENO, Euler equations, MHD equations, relativistic MHD equations.

1 Introduction and context

This paper deals with the development of a new family of arbitrary high order accurate finite volume schemes in space and time for the solution of nonlinear systems of hyperbolic partial differential equations, as for instance the Euler equations of compressible gas dynamics, or the ideal classical and relativistic magneto-hydrodynamics system (MHD/RMHD). More precisely, we propose to couple the recently developed *a posteriori* "Multi-dimensional Optimal Order Detection" (MOOD) concept [18,29,30] with the "Arbitrary high order DERivatives" (ADER) scheme, that allows us to reach arbitrary order of accuracy in space and time in one single step. The resulting method will be denoted by ADER-MOOD in the following.

By 'higher order' we strictly consider *better than second order accurate* methods, whose effective numerical order of accuracy is optimal for smooth solutions. More importantly, such numerical methods must be efficient on general unstructured meshes in multiple space dimensions. Several of such methods are available under the key names Discontinuous Galerkin (DG) methods [19, 21–24, 40], ENO, WENO and HWENO methods (Weighted/Hermite Essentially Non Oscillatory) [1,4,9,16,50,52,55,57,58,68–70,76,85,92], ADER schemes [2, 8, 9, 37–39, 59, 74, 75, 83, 84, 86, 88, 91], $P_N P_M$ schemes [32, 34, 62, 63], etc.

By efficient we understand that such methods can be implemented with acceptable effort and must be numerically validated on a test suite as exhaustive as possible; they can run on computers of reasonable size, and they must be robust, stable and accurate. This set of properties seems to be a general agreement amongst developers of numerical methods for conservation laws.

To avoid Gibbs phenomenon that occurs when shock waves or steep gradients are present, any better than first order accurate scheme must add some sort of extra dissipation usually manifested by a non linearity in the scheme. This is a direct consequence