Commun. Comput. Phys.
doi: 10.4208/cicp.150313.171013s

Vol. **15**, No. 4, pp. 1045-1067
April 2014

Immersed Finite Element Method for Interface Problems with Algebraic Multigrid Solver

Wenqiang Feng[1], Xiaoming He[1,*], Yanping Lin[2,3] and Xu Zhang[4]

[1] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409, USA.*
[2] *Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong.*
[3] *Department of Mathematical and Statistics Science, University of Alberta, Edmonton, AB, T6G 2G1, Canada.*
[4] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA.*

Received 15 March 2013; Accepted (in revised version) 17 October 2013

Available online 21 January 2014

Abstract. This article is to discuss the bilinear and linear immersed finite element (IFE) solutions generated from the algebraic multigrid solver for both stationary and moving interface problems. For the numerical methods based on finite difference formulation and a structured mesh independent of the interface, the stiffness matrix of the linear system is usually not symmetric positive-definite, which demands extra efforts to design efficient multigrid methods. On the other hand, the stiffness matrix arising from the IFE methods are naturally symmetric positive-definite. Hence the IFE-AMG algorithm is proposed to solve the linear systems of the bilinear and linear IFE methods for both stationary and moving interface problems. The numerical examples demonstrate the features of the proposed algorithms, including the optimal convergence in both L^2 and semi-H^1 norms of the IFE-AMG solutions, the high efficiency with proper choice of the components and parameters of AMG, the influence of the tolerance and the smoother type of AMG on the convergence of the IFE solutions for the interface problems, and the relationship between the cost and the moving interface location.

AMS subject classifications: 65F10, 65N30, 35J15, 35K20

Key words: Interface problems, immersed finite elements, algebraic multigrid method.

1 Introduction

In this article, we first consider the following second order elliptic interface problem:

*Corresponding author. *Email addresses:* `fw253@mst.edu` (W. Feng), `hex@mst.edu` (X. He), `yanlin@ualberta.ca` (Y. Lin), `xuz@vt.edu` (X. Zhang)

$$\begin{cases} -\nabla\cdot(\beta\nabla u)=f(X), & X\in\Omega, \\ u(X)=g(X), & X\in\partial\Omega, \end{cases} \tag{1.1}$$

together with the jump conditions on the interface Γ:

$$[u]|_\Gamma=0, \tag{1.2a}$$

$$\left[\beta\frac{\partial u}{\partial \mathbf{n}}\right]\Big|_\Gamma=0. \tag{1.2b}$$

Here, see Fig. 1, without loss of generality, we consider the case in which $\Omega\subset IR^2$ is an open rectangular domain, and the interface curve Γ is defined by a smooth function which separates Ω into two sub-domains Ω^-, Ω^+ such that $\overline{\Omega}=\overline{\Omega^-}\cup\overline{\Omega^+}\cup\Gamma$, and the coefficient $\beta(X)$ is a positive piecewise constant function defined by

$$\beta(X)=\begin{cases} \beta^-, & X\in\Omega^-, \\ \beta^+, & X\in\Omega^+. \end{cases}$$

We will also consider the following parabolic moving interface problem:

$$\begin{cases} u_t-\nabla\cdot(\beta\nabla u)=f(t,X), & X\in\Omega, \quad t\in(0,T_{end}], \\ u(t,X)=g(t,X), & X\in\partial\Omega, \quad t\in(0,T_{end}], \\ u(0,X)=u_0(X), & X\in\overline{\Omega}, \end{cases} \tag{1.3}$$

with the jump condition on a moving interface $\Gamma(t)$:

$$[u]|_{\Gamma(t)}=0, \tag{1.4a}$$

$$\left[\beta\frac{\partial u}{\partial \mathbf{n}}\right]\Big|_{\Gamma(t)}=0. \tag{1.4b}$$

Without loss of generality, we consider the case in which the interface curve $\Gamma(t)$ is defined by a smooth function $\Gamma:[0,T_{end}]\to\Omega$. At any time $t\in[0,T_{end}]$, the interface $\Gamma(t)$ separates Ω into two sub-domains $\Omega^+(t)$ and $\Omega^-(t)$ such that $\Omega=\Omega^+(t)\cup\Omega^-(t)\cup\Gamma(t)$ and $\Gamma(t)\cap\partial\Omega=\emptyset$. The coefficient function $\beta(t,X)$ is discontinuous across the interface $\Gamma(t)$. For simplicity, we assume $\beta(t,X)$ is a piece-wise constant function as follows:

$$\beta(t,X)=\begin{cases} \beta^-, & X\in\Omega^-(t), \\ \beta^+, & X\in\Omega^+(t). \end{cases}$$

The stationary interface problems (1.1)-(1.2b) and the moving interface problem (1.3)-(1.4b) are involved in many applications of engineering and sciences, such as the field injection problem [25, 77], flow problem [3, 15], electromagnetic problems [4, 8, 43], shape/toplogy optimization problem [7, 19], and the Stefan problem [11, 62]. These interface problems can be solved by conventional finite difference or finite element methods with optimal convergence if a body-fitting mesh is utilized [5, 6, 9, 14, 38]. However,