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The Stability and Convergence of Fully Discrete Galerkin-Galerkin FEMs for Porous Medium Flows

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Abstract. The paper is concerned with the unconditional stability and error estimates of fully discrete Galerkin-Galerkin FEMs for the equations of incompressible miscible flows in porous media. We prove that the optimal L^2 error estimates hold without any time-step (convergence) conditions, while all previous works require certain time-step restrictions. Theoretical analysis is based on a splitting of the error into two parts: the error from the time discretization of the PDEs and the error from the finite element discretization of the corresponding time-discrete PDEs, which was proposed in our previous work [26, 27]. Numerical results for both two and three-dimensional flow models are presented to confirm our theoretical analysis.

AMS subject classifications: 65N12, 65N30, 35K61

Key words: Unconditional stability, optimal error estimate, Galerkin FEMs, incompressible miscible flows.

1 Introduction

We consider incompressible miscible flow in porous media, which is governed by the following system of equations:

$$\Phi\frac{\partial c}{\partial t}-\nabla\cdot(D(\mathbf{u})\nabla c)+\mathbf{u}\cdot\nabla c=\hat{c}q^I-cq^P, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=q^I-q^P, \tag{1.2}$$

$$\mathbf{u}=-\frac{k(x)}{\mu(c)}\nabla p, \tag{1.3}$$

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where p is the pressure of the fluid mixture, $\mathbf{u}$ is the velocity and c is the concentration; $k(x)$ is the permeability of the medium, $\mu(c)$ is the concentration-dependent viscosity, Φ is the porosity of the medium, q^I and q^P are the given injection and production sources, $\hat{c}$ is the concentration in the injection source, and $D(\mathbf{u})=[D_{ij}(\mathbf{u})]_{d\times d}$ is the diffusion-dispersion tensor which may be given in different forms (see [5,6] for details). We assume that the system is defined in a bounded smooth domain Ω in $\mathbb{R}^d$ $(d=2,3)$, for $t\in[0,T]$, coupled with the initial and boundary conditions:

$$\mathbf{u}\cdot\mathbf{n}=0, \quad D(\mathbf{u})\nabla c\cdot\mathbf{n}=0 \qquad \text{for } x\in\partial\Omega,\ t\in[0,T], \tag{1.4a}$$

$$c(x,0)=c_0(x) \qquad \text{for } x\in\Omega. \tag{1.4b}$$

The system (1.1)-(1.4) has been studied extensively in the last several decades, see [11, 35] and the references therein. Existence of weak solutions of the system was obtained by Feng [20] for the 2D model and by Chen and Ewing [9] for the 3D problem. Existence of semi-classical/classical solutions is unknown. Numerical simulations have been done with various applications [4, 7, 13, 17, 40, 41]. Optimal error estimates of a Galerkin-Galerkin method for the system in two-dimensional space was given first by Ewing and Wheeler [18] roughly under the time-step condition $\tau=o(h)$, in which a linearized semi-implicit Euler scheme was used in the time direction and a standard Galerkin FE approximation was used for both the concentration and the pressure. Later, a Galerkin-mixed finite element method was proposed by Douglas et al. [12] for this system, where a Galerkin approximation was applied for the concentration equation and a mixed approximation in the Raviart-Thomas finite element space [38] was used for the pressure equation. A linearized semi-implicit Euler scheme, the same as one used in [18], was applied for the time discretization. Optimal error estimates were obtained under a similar time-step condition $\tau=o(h)$. There are many other numerical methods in the literature for solving the equations of incompressible miscible flows in porous media, such as see [46] for an ELLAM in two-dimensional space, [47] for an MMOC-MFEM approximation for the 2D problem, [14, 43] for a characteristic-mixed method in two and three dimensional spaces, respectively, and [30, 31] for a collocation-mixed method and a characteristic-collocation method, respectively. In all those works, error estimates were established under certain time-step conditions. Moreover, it has been noted that linearized semi-implicit schemes have been analyzed for many other nonlinear parabolic-type systems, such as the Navier-Stokes equations [2, 19, 21, 24, 28], nonlinear thermistor problems [15, 51], viscoelastic fluid flow [8, 16, 48], KdV equations [33, 50], nonlinear Schrödinger equation [3, 39, 45], Ginzburg-Landau equations [10, 29] and some other equations [22, 42]. A time-step condition was always imposed to get suitable error estimates. A key issue in analysis of FEMs is the boundedness of the numerical solution in L^∞ norm or a stronger norm, which in a routine way can be estimated by mathematical induction with an inverse inequality, such as,

$$\|u_h^n-R_hu(\cdot,t_n)\|_{L^\infty}\le Ch^{-d/2}\|u_h^n-R_hu(\cdot,t_n)\|_{L^2}\le Ch^{-d/2}(\tau^m+h^{r+1}), \tag{1.5}$$