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Numerical Solution of Blow-Up Problems for Nonlinear Wave Equations on Unbounded Domains

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Abstract. The numerical solution of blow-up problems for nonlinear wave equations on unbounded spatial domains is considered. Applying the unified approach, which is based on the operator splitting method, we construct the efficient nonlinear local absorbing boundary conditions for the nonlinear wave equation, and reduce the nonlinear problem on the unbounded spatial domain to an initial-boundary-value problem on a bounded domain. Then the finite difference method is used to solve the reduced problem on the bounded computational domain. Finally, a broad range of numerical examples are given to demonstrate the effectiveness and accuracy of our method, and some interesting propagation and behaviors of the blow-up problems for nonlinear wave equations are observed.

AMS subject classifications: 65M06, 65M20

Key words: Finite-time blow-up, nonlinear wave equation, absorbing boundary conditions, finite difference method, unbounded domains.

1 Introduction

This paper is devoted to studying the numerical solution of blow-up problems for the nonlinear wave equation of the form

$$u_{tt}=a^2\Delta u+|u|^p, \quad x\in\Omega, \quad t>0, \tag{1.1}$$

$$u(x,0)=\phi_0(x), \quad u_t(x,0)=\phi_1(x), \quad x\in\Omega, \tag{1.2}$$

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where $u(x,t)$ represents the wave displacement at position x and time t, Δ denotes the Laplacian, a is the given reference wave speed, $\phi_0(x)$ and $\phi_1(x)$ are the initial values, and the spatial domain Ω is given by $\Omega=\mathbb{R}^d$ $(d=1,2,\cdots)$. The wave equation appears in applications in various areas of mathematical physics. As an example, the blow-up problem would be that of the focusing energy-subcritical nonlinear wave equations with electromagnetic potential in electromagnetism [1] (For more applications see the list of the references in [2–4]).

The theory of finite-time blow-up for nonlinear wave equations has an interesting and exciting history. We will only give a brief summary and refer the reader to the papers [5–11] and the survey paper [12]. Strauss [5] conjectured that if the critical value, $p_0(d)$, is the positive root of $(d-1)x^2-(d+1)x-2=0$, then, if $1<p<p_0(d)$, the solution of the nonlinear wave equation blows up in finite time for any choice of the initial conditions. Glassey [6, 7] subsequently verified the conjecture in two dimensions by showing that $p_0(2)=\frac{1}{2}(3+\sqrt{17})$. Sideris proved in his PhD thesis that there exists blow-up in finite time for all $p>1$ when $d=1$, and he also proved in [8] the conjecture of Strauss in high dimensions by averaging the Riemann function in time. The papers [9–11] contain a systematic analysis of the life-span of classical solutions for nonlinear wave equations.

For semilinear parabolic PDEs (reaction-diffusion equations) arising as models in combustion theory, the theoretical and physical aspects of single-point blow-up versus total blow-up are well understood (see for example Bebernes and Eberly [13], Section 5.5, and Lacey [14]). However, analogous studies on the possible sets of blow-up points for nonlinear wave equations do to the best of our knowledge not exist.

The numerical analysis of the blow-up problems for nonlinear wave equations (1.1)-(1.2) has so far received little attention, see [15, 16] and their references. For the bounded computational domain case, there are only few papers studying the numerical solution of blow-up problems. Cho [4] gave a finite difference scheme and proposed a rule for time-stepping for blow-up solutions of the one-dimensional nonlinear wave equation. For unbounded computational domains, the difficulties in the numerical solution of the problem (1.1)-(1.2) include three parts: the nonlinearity, the unboundedness, and the multidimensionality. To deal with the nonlinearity, we use the idea of the unified approach which was introduced by Zhang *et al.* [17, 18]; the basic idea underlying the unified approach is the well-known time-splitting method. Xu *et al.* [19] designed absorbing boundary conditions for nonlinear Schrödinger equations by using the time-splitting method. The idea of time-splitting method was extended for solving two-dimensional sine-Gordon equation by Han *et al.* [20]. For unboundedness, one of the most popular approaches is the use of the absorbing boundary conditions (ABCs) method, which is a powerful method to reduce the problems on an unbounded domain to a bounded domain, for an appropriate bounded computational domain. For the multidimensional case, it is hard to find suitable absorbing boundary conditions at the corners of a rectangle; we construct the conditions of corners by the average of two artificial boundary conditions to overcome this difficulty.

How to select a suitable bounded computational domain and derive appropriate ab-