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An Analog of Einstein's General Relativity Emerging from Classical Finite Elasticity Theory: Analytical and Computational Issues

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Abstract. The "analogue gravity formalism", an interdisciplinary theoretical scheme developed in the past for studying several non relativistic classical and quantum systems through effective relativistic curved space-times, is here applied to largely deformable elastic bodies described by the nonlinear theory of solid mechanics. Assuming the simplest nonlinear constitutive relation for the elastic material given by a Kirchhoff-St Venant strain-energy density function, it is possible to write for the perturbations an effective space-time metric if the deformation is purely longitudinal and depends on one spatial coordinate only. Theoretical and numerical studies of the corresponding dynamics are performed in selected cases and physical implications of the results obtained are finally discussed.

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1 Introduction

The mathematical structure behind Einstein's General Relativity (GR) is differential geometry. GR is a physical theory whose foundations lay in elegant variational principles for geometry and matter fields. Recently it has been found an analog of relativistic gravity manifested by several systems of non-relativistic condensed matter physics, mainly

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in the fields of classical and quantum fluids and in electromagnetism [1–3]. The origin of such an analogy has to be searched in the classical d'Alembert wave equation

$$\frac{\partial^2}{\partial t^2}\xi - v^2\nabla^2\xi = 0 \tag{1.1}$$

($\nabla^2 \equiv \Delta$ is the ordinary Laplacian operator in Euclidean space) which describes the propagation at constant speed v of a certain quantity ξ in an homogeneous and isotropic background medium or even in vacuum if v is the speed of light (so that such an equation results Lorentz invariant). As an example, in fluid dynamics, the quantity ξ describes the perturbations of pressure or of the velocity potential, in infinitesimal elasticity it appears as a transverse or longitudinal relative displacement vector and in electromagnetism it stands for the electric or magnetic field vectors. If the medium is not homogeneous, the corresponding wave equations become more complicated. The mathematical relations involved in fact contain now second order terms with mixed derivatives together with first order ones, all of these multiplied by time and/or space dependent coefficients. The equations for small elastic waves in anisotropic and inhomogeneous media are a typical example of such a situation [4, 5]. Because all the aforementioned mathematical expressions resemble very much the structure of second order wave equations in curved GR spacetimes, the question in the past naturally arose whether a possible connection between all of these Newtonian and Relativistic problems could be found. The first confirmation of such an hypothesis historically dates back to Unruh's work [6, 7] on perfect fluid perturbations, later extended by Visser and collaborators [8, 9]. In short, for a non relativistic, classical, perfect, irrotational, compressible and barotropic fluid, its linear perturbations can be rewritten as a real massless scalar field equation on a curved space-time characterized by an acoustic four dimensional metric tensor. Recently the authors have shown [10] the connection of the theory described above with the quasi-linear decoupled second order wave equation governing the velocity potential as derived by Von Mises [11]. Several complementary studies have been performed in the last decade then, focusing in particular on analogs of black/white holes and cosmological systems [12–28, 30]. The aim of this article is to clarify if the Analogue Gravity formulation just discussed could be applied also to the theory of nonlinear elasticity. Infinitesimal elasticity is a limiting case of the much more complicated theory of nonlinear solid mechanics, whose natural language is differential geometry. An appropriate introduction to linear (infinitesimal) elasticity can be found in Feynman's Lectures on Physics books [31] or in volume 7 of Landau-Lifshitz' Theoretical Physics course [5]. For the purposes of our study, in Section 2 we shall here introduce a short resume of nonlinear solid mechanics accounting however for appropriate references for this topic. The unknown quantities in continuum mechanics are the components of the relative displacement vector $\vec{u}$, a Lagrangian entity intrinsically three dimensional which maps the position of a material point initially located in $\vec{x}$ into a new point $\vec{x}' \equiv \vec{x} + \vec{u}$. A Lagrangian point of view is here adopted because in solid mechanics one deals with the complicated free boundary problem of locating the body's surface during the deformation [32], although an Eulerian