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THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR OPTIMAL CONTROL PROBLEM GOVERNED BY CONVECTION DIFFUSION EQUATIONS

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Abstract. In this paper we analyze the Local Discontinuous Galerkin (LDG) method for the constrained optimal control problem governed by the unsteady convection diffusion equations. A priori error estimates are obtained for both the state, the adjoint state and the control. For the discretization of the control we discuss two different approaches which have been used for elliptic optimal control problem.

Key Words. Local Discontinuous Galerkin method, unsteady convection diffusion equations, constrained optimal control problem, a priori error estimate.

1. Introduction

In this paper, we consider the following linear-quadratic optimal control problems for state variable y and the control variable u involving pointwise control constraints:

$$\min_{u\in K\subset X}\left\{\frac{1}{2}\int_0^T\int_\Omega (y(x,t)-y_d(x,t))^2dxdt+\frac{\alpha}{2}\int_0^T\int_{\Omega_U}u(x,t)^2dxdt\right\} \tag{1}$$

subject to

$$\begin{aligned} y_t+\nabla\cdot(\vec{\beta}y-\varepsilon\nabla y)&=f+\mathcal{B}u, \quad x\in\Omega,\quad t\in(0,T],\\ (\vec{\beta}y-\varepsilon\nabla y)\cdot\vec{n}&=\tilde{y} \qquad \text{on} \quad \partial\Omega_I,\\ \varepsilon\nabla y\cdot\vec{n}&=0 \qquad \text{on} \quad \partial\Omega_O,\\ y(x,0)&=y_0(x), \qquad x\in\Omega. \end{aligned} \tag{2}$$

Here Ω and Ω_U are bounded open sets in R^2 with boundaries $\partial\Omega$ and $\partial\Omega_U$; $K\subset X$ is bounded convex set. The details will be specified in the next section.

Although the a priori error estimates for finite element discretization of optimal control problem governed by elliptic equations and parabolic equations have been discussed in many publications, see, e.g., [1], [7], [13], [16], there are very few results on the a priori error estimates of optimal control problem governed by convection diffusion equations. Some related work can be find in, e.g., [2], [3], [5], [18].

In the optimal control problem (1)-(2), the state equation is a convection diffusion equation. It is well known that the standard finite element discretizations applied to the convection diffusion problem (2) lead to strong oscillation when ε is small. There are some effective discretization schemes which are introduced

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to improve the approximation properties of standard Galerkin method and to reduce the oscillatory behavior, see, e.g., [4], [11], [12]. Recently, a new discretization scheme was proposed in [6] for the convection diffusion equation, which is called Local Discontinuous Galerkin method. The analysis of Local Discontinuous Galerkin method has been extended to many equations, such as, elliptic equation, nonlinear convection diffusion equation, oseen equations and stokes equations .

In this paper, we use the Local Discontinuous Galerkin method to approximate the state equation in the optimal control problem (1)-(2). For the control discretization we discussed two different methods. The first is the classic finite element discretization. The control variable is discretized by piecewise constant and piecewise linear finite element spaces, respectively. The second is a variational approach proposed in [10], where no explicit discretization of the control variable is used and the discrete control variable is achieved by projecting the discrete adjoint state variable on the admissible control set. For above LDG scheme, a priori error estimates of the semi-discrete and fully-discrete approximation schemes for the state, the adjoint state and the control are derived. To our best knowledge, the similar results has not yet been reported in the open literature.

This paper is organized as follows: In Section 2, we introduce the model problem for the optimal control problem governed by the unsteady convection diffusion equations and present the LDG approximation scheme of the model problem. In Section 3, we prove a priori error estimate of the semi-discretization scheme for the optimal control problem. In Section 4, a priori error estimate of the full discretization scheme for the optimal control problem is derived. In the last section, we briefly summarize the method used, the results obtained and possible future extensions and challenges.

2. LDG scheme for the optimal control problem

Let us introduce some standard notations. We adopt the notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω, with a norm $\|\cdot\|_{m,q,\Omega}$ and a semi-norm $|\cdot|_{m,q,\Omega}$. For q=2, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$. Furthermore, we set $W_0^{1,q}(\Omega) = \{v \in W^{1,q}(\Omega) : \gamma v \mid_{\partial\Omega} = 0\}$, where γv is the trace of v on the boundary $\partial\Omega$. The inner products in $L^2(\Omega_U)$ and $L^2(\Omega)$ are indicated by $(\cdot,\cdot)_U$ and $(\cdot,\cdot)$, respectively. For $p \in [1,\infty)$, the internal $[0,T] \subset R$ and the Banach space A with norm $\|\cdot\|_A$, we denote by $L^p(0,T;A)$ the set of measurable functions $y : [0,T] \to A$ such that $\int_0^T \| y \|_A^p \, dt \leq \infty$. The norm on $L^p(0,T;A)$ is defined by

$$\| y(t) \|_{L^p(0,T;A)} = \begin{cases} (\int_0^T \| y(t) \|_A^p \, dt)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ ess \sup\limits_{t\in[0,T]} \| y(t) \|_A, & p = \infty. \end{cases}$$

In addition c and C denote generic constants.

In this section we provide a numerical scheme to approximate the distributed convex optimal control problem governed by evolutionary convection diffusion equations. We shall take the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$ to fix the idea.

Consider the following constrained optimal control problem governed by evolutionary convection diffusion equations:

$$(3) \qquad \min_{u\in K\subset X} \left\{ \frac{1}{2}\int_0^T \int_\Omega (y(x,t) - y_d(x,t))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_{\Omega_U} u(x,t)^2 dxdt \right\}$$