

## A Charge Preserving Scheme for the Numerical Resolution of the Vlasov-Ampère Equations

Nicolas Crouseilles<sup>1,\*</sup> and Thomas Respaud<sup>2</sup>

<sup>1</sup> INRIA-Nancy-Grand Est, CALVI Project, Strasbourg, France.

<sup>2</sup> IRMA, Université de Strasbourg and INRIA-Nancy-Grand Est, CALVI Project, Strasbourg, France.

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**Abstract.** In this report, a charge preserving numerical resolution of the 1D Vlasov-Ampère equation is achieved, with a forward Semi-Lagrangian method introduced in [10]. The Vlasov equation belongs to the kinetic way of simulating plasmas evolution, and is coupled with the Poisson's equation, or equivalently under charge conservation, the Ampère's one, which self-consistently rules the electric field evolution. In order to ensure having proper physical solutions, it is necessary that the scheme preserves charge numerically. *B*-spline deposition will be used for the interpolation step. The solving of the characteristics will be made with a Runge-Kutta 2 method and with a Cauchy-Kovalevsky procedure.

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**Key words:** Semi-Lagrangian method, charge conservation, Runge-Kutta, Cauchy-Kovalevsky, *B*-spline deposition.

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## 1 Introduction

In order to describe the dynamics of charged particles in a plasma or in a propagating beam, the Vlasov equation can be used to calculate the response of the plasma to the electromagnetic fields. The unknown  $f(t, x, v)$  which depends on the time  $t$ , the space  $x$  and the velocity  $v$  represents the distribution function of the studied particles. The coupling with the self-consistent electromagnetic fields is taken into account through the Maxwell's equations.

The numerical solution of such systems is most of the time performed using Particle In Cell (PIC) methods, in which the plasma is approximated by macro-particles (see [3]). They are advanced in time with the electromagnetic fields which are computed on a grid.

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\*Corresponding author. *Email address:* crouseil@math.u-strasbg.fr (N. Crouseilles)

However, despite their capability to treat complex problems, PIC methods are inherently noisy. This becomes problematic when low density or highly turbulent regions are studied. Hence, numerical methods which discretize the Vlasov equation on a grid of the phase space can offer a good alternative to PIC methods (see [6,7,12,16,17,28,34]).

An important issue for electromagnetic PIC or Vlasov solvers in which the fields are computed through the Maxwell's equations, is the problem of discrete charge conservation. The electric and magnetic fields via Ampère and Faraday's equations have to be computed in such a way that it satisfies a discrete Gauss law at each time step. Indeed, the charge and current densities computed from the particles (for PIC methods) or from  $f$  (for Vlasov methods) do not verify the continuity equation so that the Maxwell's equations with these sources might be ill-posed.

Two main issues have been explored in the literature, mainly in the PIC context. The first one consists in modifying the inconsistent electric field resulting from an ill-posed Maxwell solver (see [24,25]). In the second approach, the current is computed in a specific way so as to enforce a discrete continuity equation (see [1,15,35,36]). The second class of methods has the advantage of being local and does not modify the electromagnetic field away from the source, which can generate errors in some applications.

More recently, Sircombe and Arber managed to create a 4D Vlasov-Maxwell charge preserving scheme using a split Eulerian approach (VALIS code described in [30]). They take full benefits of conservative methods computing the current using the fluxes in space after each spatial advection. In addition to some specific properties that can be controlled through filters (positivity, monotonicity), conservative methods applied to multi-dimensional problems can be solved through the succession of unidimensional problems thanks to a directional splitting like for instance in [27]. Thus, conservative methods have proven very efficient for the solving of transport equations (for details see [8,9,16,17,38]). In this work, we also deal with a phase space grid to simulate the Vlasov equation, but using the Forward semi-Lagrangian method (FSL detailed in [10]); as explained in [10], since FSL bears similarities with PIC methods, we shall extend the approach of [1,13,36] to our FSL context. In a few words, the time-dependent current is averaged over the one-step trajectory of the particles (which correspond to the grid points in FSL).

The FSL approach has been developed on a cubic  $B$ -splines reconstruction. Different time algorithms are proposed to solve the characteristics of the Vlasov equations: Runge-Kutta or Cauchy-Kovaleskaya algorithms. As explained in [10] the main advantages of the FSL method are: (i) it is conservative (thanks to the partition of the unity), (ii) it can be easily extended to arbitrary high order time algorithms in its unsplit form (using CK algorithms), and (iii) it is equivalent to the BSL counterpart when 1D constant advections are considered.

The coupling of FSL with charge preserving algorithms is the main goal of this work. We focus on the 1D Vlasov-Ampère and quasi-relativistic Vlasov-Maxwell models to show the feasibility and the advantages of the approach. The extension to the 2D Vlasov-Maxwell case will be the object of a future work in which we hope to present an alternative approach to the VALIS one presented in [30]. Here, we will see that if a discrete