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An Adjoint State Method for Numerical Approximation of Continuous Traffic Congestion Equilibria

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Abstract. The equilibrium metric for minimizing a continuous congested traffic model is the solution of a variational problem involving geodesic distances. The continuous equilibrium metric and its associated variational problem are closely related to the classical discrete Wardrop's equilibrium. We propose an adjoint state method to numerically approximate continuous traffic congestion equilibria through the continuous formulation. The method formally derives an adjoint state equation to compute the gradient descent direction so as to minimize a nonlinear functional involving the equilibrium metric and the resulting geodesic distances. The geodesic distance needed for the state equation is computed by solving a factored eikonal equation, and the adjoint state equation is solved by a fast sweeping method. Numerical examples demonstrate that the proposed adjoint state method produces desired equilibrium metrics and outperforms the subgradient marching method for computing such equilibrium metrics.

AMS subject classifications: 65K10, 90C25, 49L25, 65N06

Key words: Traffic congestion, adjoint state method, gradient descent, fast sweeping method, factored eikonal equation.

1 Introduction

In traffic flow for transportation and communication, network equilibrium models are commonly used for prediction of traffic patterns in transportation and communication networks that are subject to congestion. The idea of traffic equilibrium originated as early as 1924 in the work by Knight [12]. In 1952, Wardrop introduced two principles that

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formalize the notion of equilibrium [24]. Wardrop's first principle states that no driver may lower his/her transportation cost through unilateral action, which leads to the user-optimized equilibrium. Wardrop's second principle states that drivers behave cooperatively to minimize the total system travel time, which leads to the system-optimized equilibrium. These two principles have been put into firm foundation by treating the network equilibrium problem as a discrete convex programming problem in Chapter 3 of Beckmann et al. [2]. In a recent work [7] Carlier et al. introduced a continuous version of Wardrop's equilibria, proved the existence of continuous traffic congestion equilibrium by introducing a variational problem analogous to the discrete convex programming in Chapter 3 of [2], and related it to the optimal transportation problem with congestion. It turns out that such an equilibrium is linked to a certain metric, and all actually used paths (the continuous version of routes) must be geodesics for this metric. Based on the work in [7], Benmansour et al. [4] have shown that an equilibrium metric is the solution of a variational problem involving geodesic distances. Furthermore, to solve this particular variational problem, they have designed a subgradient marching method [3] to approximate continuous traffic congestion equilibria. This method requires intensive memory and is computationally inefficient.

In this paper, as an alternative approach, we propose a new adjoint state method which is efficient in both memory and computation to solve this variational problem. By using this adjoint state method, we first derive the gradient descent direction for a certain nonlinear functional in a continuous setting, and we then discretize the resulting gradient accordingly. This is different from the viewpoint of Benmansour et al. [4], where they discretized the nonlinear functional first and computed the derivatives of the discrete functional with respect to metrics in a discrete setting. In designing an efficient adjoint state method for continuous traffic congestion equilibria, there are two challenging computational issues: one is how to compute geodesic distances efficiently and accurately from a source location to many destination locations, as the distance function is not differentiable at the source location; the other is how to solve the adjoint state equation efficiently. To deal with the first difficulty we use the factored eikonal equation [11] to discretize the eikonal equation so that the source singularity can be treated with high accuracy. To deal with the second difficulty, we adopt a fast sweeping method as designed in [13]. Numerical examples demonstrate that the proposed adjoint state method produces desired equilibrium metrics. In a discrete setting with N grid points, in terms of computational memory, the subgradient marching method in [3, 4] requires $\mathcal{O}(N^2)$ memory, while our new approach requires $\mathcal{O}(N)$ memory; in terms of computational complexity, the subgradient marching method in [3,4] is of $\mathcal{O}(N^2 \log N)$, while our approach is of $\mathcal{O}(N)$.

The outline of this work is as follows. In Section 2, we present the continuous traffic congestion model and its dual formulation as described in [4,7]. In Section 3, we derive our adjoint state method to compute the equilibrium metric. In Section 4, we present numerical examples to illustrate the performance of our method and make comparison with the subgradient marching method. We conclude the paper with some remarks.