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Exponential Compact Higher Order Scheme for Nonlinear Steady Convection-Diffusion Equations

Y. V. S. S. Sanyasiraju* and Nachiketa Mishra

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India.

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Abstract. This paper presents an exponential compact higher order scheme for Convection-Diffusion Equations (CDE) with variable and nonlinear convection coefficients. The scheme is $\mathcal{O}(h^4)$ for one-dimensional problems and produces a tri-diagonal system of equations which can be solved efficiently using Thomas algorithm. For two-dimensional problems, the scheme produces an $\mathcal{O}(h^4+k^4)$ accuracy over a compact nine point stencil which can be solved using any line iterative approach with alternate direction implicit procedure. The convergence of the iterative procedure is guaranteed as the coefficient matrix of the developed scheme satisfies the conditions required to be positive. Wave number analysis has been carried out to establish that the scheme is comparable in accuracy with spectral methods. The higher order accuracy and better rate of convergence of the developed scheme have been demonstrated by solving numerous model problems for one and two-dimensional CDE, where the solutions have the sharp gradient at the solution boundary.

AMS subject classifications: 65N05, 76M20, 35Q35

Key words: Finite difference, higher order, exponential compact, convection-diffusion.

1 Introduction

Convection-diffusion equation appears in the modelling of different fluid flow problems like contaminant transport and heat/mass transport etc. The traditional Finite Difference (FD) schemes such as central difference and upwind schemes on uniform grids have certain drawbacks viz. stability and/or accuracy, especially for convection dominated problems. Very large number of grid points are required to overcome such drawbacks and to obtain any reliable solution. In the last few years, Higher Order Compact (HOC) FD schemes have received great attention for solving CDE, which are computationally

*Corresponding author. *Email addresses:* `sryedida@iitm.ac.in` (Y. V. S. S. Sanyasiraju), `mishra.nachiketa@gmail.com` (N. Mishra)

efficient. All most all HOC schemes are either polynomial [1, 6, 8, 12, 14] or exponential type. For convection dominated problems polynomial schemes are less efficient due to the failure of discrete maximum principle and/or upwind effect in the scheme.

The finite difference methods, whose coefficients involve exponential functions of the coefficients of the corresponding differential operator and mesh width are known as exponentially fitted FD schemes [11]. HOC exponential schemes are the higher order compact exponentially fitted schemes. In the exponentially fitted FD schemes, the exponential function helps in introducing an artificial diffusion that preserves the upwind effect. Its coefficient matrix is unconditionally diagonally dominant. Such schemes are very suitable to solve the singularly perturbed convection-diffusion equations. Exponential FD schemes are first introduced by Allen and Southwell [2] to solve second order partial differential equations governing the transport of vorticity. Later, modified and improved schemes along with some analysis on their applicability are proposed in [4, 5, 10, 13, 15]. The more efficient as well as accurate scheme among the existing exponential HOC schemes is the SRECHOS [13]. So far SRECHOS has been developed for convection-diffusion equations with constant convection coefficients. Since most of the partial differential equations of practical importance have variable or nonlinear convection coefficients, therefore, the purpose of the present article is to develop and validate SRECHOS like scheme to CDE with variable and nonlinear convection coefficients. The development of the scheme is more or less in the lines of [15], however the final scheme differs from [15] as the present scheme does not require second-order derivatives of the source function in its implementation which will help in a substantial reduction in the CPU times and a better accuracy in some cases.

If SRECHOS [13] is applied directly to the CDE with variable convection coefficients, it will be of second order accurate. Using the method of modified equations, fourth order accurate scheme for CDE has been proposed in Section 2. The iterative scheme for the nonlinear CDE has been presented in Section 3. The positivity of the scheme, which is crucial for the convergence of the iterative scheme in solving the nonlinear CDE, has been shown in Section 4. The spectral analysis is also been presented in this section. Finally, all the schemes are experimentally verified with the existing schemes in Section 5 for variable, nonlinear and coupled nonlinear CDE.

2 Development of the scheme

Consider the two-dimensional nonlinear steady convection-diffusion equation

$$-au_{xx}-bu_{yy}+c(x,y,u)u_x+d(x,y,u)u_y=f(x,y), \tag{2.1}$$

on $\Omega\subset\mathbb{R}^2$, with boundary condition

$$u(x,y)=g(x,y), \quad \text{on } \partial\Omega, \tag{2.2}$$

where $a,b>0$ are constant diffusion, c,d are convection coefficients and f, g are sufficiently smooth functions with respect to x and y. If $0<a,b<1$ are very small when compared