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Trigonometric WENO Schemes for Hyperbolic Conservation Laws and Highly Oscillatory Problems

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Abstract. In this paper, we use trigonometric polynomial reconstruction, instead of algebraic polynomial reconstruction, as building blocks for the weighted essentially non-oscillatory (WENO) finite difference schemes to solve hyperbolic conservation laws and highly oscillatory problems. The goal is to obtain robust and high order accurate solutions in smooth regions, and sharp and non-oscillatory shock transitions. Numerical results are provided to illustrate the behavior of the proposed schemes.

AMS subject classifications: 65M06, 65M99, 35L65

Key words: TWENO scheme, hyperbolic conservation laws, highly oscillatory problem, finite difference scheme.

1 Introduction

In this paper, we investigate using trigonometric polynomial reconstruction as building blocks for the weighted essentially non-oscillatory (WENO) finite difference schemes, termed as TWENO schemes, to solve hyperbolic conservation laws:

$$\begin{cases} u_t+f_x(u)=0, \\ u(x,0)=u_0(x). \end{cases} \tag{1.1}$$

Hyperbolic conservation laws appear often in applications, such as in gas dynamics and modelling of shallow waters, among many others. As a result, devising robust, accurate and efficient methods for numerically solving these problems is of considerable importance and has attracted the interest of many researchers and practitioners. In 1959,

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Godunov [4] proposed a first-order numerical scheme for solving hyperbolic conservation laws. In order to achieve uniform high order of accuracy, Harten and Osher [6] gave a weaker version of the TVD (Total Variation Diminishing) [5] criterion, and also obtained the essentially non-oscillatory (ENO) type schemes. The key idea of ENO schemes is to apply the most smooth stencil among all candidate stencils to approximate the variables at cell boundaries to a high order of accuracy and to avoid oscillations near discontinuities. In 1994, Liu et al. [9] proposed a Weighted ENO (WENO) scheme that was constructed from the *r-th* order ENO schemes to obtain $(r+1)$-*th* order accuracy. Then in 1996, Jiang and Shu [7] proposed a framework to construct finite difference WENO schemes from the *r-th* order (in L^1-norm sense) ENO schemes to get $(2r-1)$-*th* order accuracy. It gave a new way of measuring the smoothness indicators and emulated the ideas of minimizing the total variation of the approximation in [14]. Recently, Zhang and Shu [16] constructed the third-order WENO scheme on three-dimensional tetrahedral meshes. A key idea in WENO schemes is the exploitation of a linear combination of lower order fluxes or reconstruction to obtain a higher order approximation. Both ENO and WENO schemes use the idea of adaptive stencils to automatically achieve high order accuracy and non-oscillatory property near discontinuities. For the system case, WENO schemes are based on local characteristic decompositions and flux splitting to avoid spurious oscillations. They have been used successfully in many applications, especially for problems containing shocks and/or complicated smooth solution structures, such as compressible turbulence simulations and aeroacoustics.

Wave-like phenomena or highly oscillatory problems are encountered quite often in nature. However, little attention has been paid to trigonometric essentially non-oscillatory schemes, which appear to be more suitable for the simulation of such problems. In 1976, Baron [1] studied trigonometric interpolation and presented certain Neville-like algorithms. Muhlbach [10–13] studied general basis function, including trigonometric interpolation in Newton form. However, their work cannot be applied to ENO type schemes directly. The methodology used cannot obey the rule of adding one interpolation point to the stencil once a time but two. Christofi [3] provided a new trigonometric reconstruction methodology that can add interpolation points one by one and can also possess necessary symmetries to be used in ENO schemes.

In this paper, following the ideas in [3, 7, 14], we construct a kind of finite difference TWENO scheme which is of 5-*th* order accurate. The main differences between [3] and this work are the way of measuring the smoothness of the trigonometric polynomials and the form of the reconstruction that the schemes are ultimately based on. A Newton form is employed in [3], but a Lagrange form used in this work that seems suitable for improving the WENO type schemes. Compared to [3], the new scheme with the same stencils can achieve an even higher order of accuracy in smooth regions and less oscillations in discontinuous regions. Compared to the WENO schemes of [7] and [14], one major advantage of the new TWENO scheme is its good performance for the wave-like and highly oscillatory problems.

The organization of this paper is as follows. In Section 2, we review and construct