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MORTAR FINITE ELEMENTS FOR COUPLING COMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS IN ELASTICITY

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Abstract. We consider the coupling of compressible and nearly incompressible materials within the framework of mortar methods. Taking into account the locking effect, we use a suitable discretization for the nearly incompressible material and work with a standard conforming discretization elsewhere. The coupling of different discretization schemes in different subdomains are handled by flexible mortar techniques. A priori error analysis is carried out for the coupled problem, and several numerical examples are presented. Using dual Lagrange multipliers, the Lagrange multipliers can easily be eliminated by local static condensation.

Key Words. Mortar finite elements, Lagrange multipliers, dual space, non-matching triangulations, mixed formulations, saddle point problems

1. Introduction

Often coupled problems with completely different material properties in different subdomains occur in solid mechanics. To get optimal a priori estimates, a proper discretization scheme should be used in each subdomain. Here, we consider coupling of compressible and nearly incompressible linear elastic materials with mortar techniques. The boundary value problem of elasticity involves a critical Lamé parameter λ. For nearly incompressible materials the Lamé parameter λ is very large, and it is well-known that working with low order finite elements with displacement based formulation suffers from so-called locking effect yielding a poor convergence, see [13, 18, 4]. Various approaches have been proposed to overcome this difficulty. Among these are to apply higher-order finite elements with a standard displacement formulation. For example, in [31], it is shown that working with the h-version finite elements of order higher than three on a class of triangular meshes completely avoid locking. On the other hand, in [4], it has been shown that the h-version can never be fully free of locking in rectangular meshes no matter how higher-order finite elements are used in the sense that optimal orders of convergence are not obtained. The other approach is related to working with mixed methods. The linear elasticity problem can be formulated as a mixed formulation in many different ways, see [18, 13, 11, 33, 3, 1]. The general approach in these mixed formulations is to introduce extra variables leading to a problem of saddle point type with a penalty term. The essential point is to prove that the method is robust for the limiting problem, which is the Stokes problem. Methods associated with nonconforming finite elements have also been analyzed leading to the uniform convergence in the nearly

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incompressible case, see [21, 17, 30, 15]. The central point in these approaches is to construct an interpolation operator at each element which preserves zero divergence. We point out that many different methods like the reduced integration, the enhanced assumed strain and the mixed enhanced strain can be analyzed within the framework of mixed formulation, see [18, 12, 14, 32, 26, 27, 29]. All these approaches have in common that the finite element approximation is robust for nearly incompressible materials.

In order to avoid the problem of locking-effect, we consider suitable discretization schemes for nearly incompressible materials. Introducing the pressure as an additional unknown for the nearly incompressible case, we arrive at the problem of coupling a saddle point problem with a positive definite one. Working exclusively with non-matching triangulations, we use mortar techniques to realize the coupling of different discretization schemes.

This paper is organized as follows. In the next section, we describe the boundary value problem of linear elasticity and introduce a new formulation of the boundary value problem in the continuous setting suitable for coupling a nearly incompressible material with a compressible material. In Section 3, we show the stability of the scheme and prove optimal a priori estimates. Finally in Section 4, we present some numerical results illustrating the performance of our approach.

2. The problem of linear elasticity in the mortar framework

We consider a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, which is decomposed into two non-overlapping subdomains Ω_1 and Ω_2 with the common interior interface Γ, $\bar{\Gamma} = \partial\Omega_1 \cap \partial\Omega_2$. For simplicity, we restrict ourselves to the case of two subdomains. However, the approach can easily be generalized to more than two subdomains.

We assume that the subdomains Ω_1 and Ω_2 are occupied with different isotropic linear elastic materials. Furthermore, the material in Ω_1 is supposed to be nearly incompressible, whereas Ω_2 is occupied with a compressible material. We consider the following linear elasticity problem of finding the displacement field $\mathbf{u}$ in Ω such that

$$\begin{aligned} -\mathrm{div}\,(\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u})) &= \boldsymbol{f}_1 \quad \text{in} \quad \Omega_1, \\ -\mathrm{div}\,(\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u})) &= \boldsymbol{f}_2 \quad \text{in} \quad \Omega_2 \end{aligned} \tag{1}$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Here, $\mathcal{C}_1$ and $\mathcal{C}_2$ are constant and symmetric fourth-order elasticity tensors corresponding to different materials in Ω_1 and Ω_2, respectively. Denoting the identity tensor by $\mathbf{1}$, their actions on the strain tensor are defined as

$$\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u}) = \lambda_1(\mathrm{div}\,\mathbf{u})\mathbf{1} + 2\mu_1\,\boldsymbol{\varepsilon}(\mathbf{u}), \text{ and } \mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}) = \lambda_2(\mathrm{div}\,\mathbf{u})\mathbf{1} + 2\mu_2\,\boldsymbol{\varepsilon}(\mathbf{u}).$$

Moreover, the plane strain is assumed in the two-dimensional case. We define the global Hooke tensor $\mathcal{C}$ which takes the value $\mathcal{C}_1$ on Ω_1 and $\mathcal{C}_2$ on Ω_2, and set $\mathbf{u}_1 := \mathbf{u}_{|\Omega_1}$ and $\mathbf{u}_2 := \mathbf{u}_{|\Omega_2}$. We assume that $\boldsymbol{f}_i \in (L^2(\Omega_i))^d, i = 1, 2$. The interface conditions on Γ are given by

$$\begin{aligned} [\mathbf{u}] &:= \mathbf{u}_1 - \mathbf{u}_2 = 0 \quad \text{on} \quad \Gamma, \\ [\mathbf{u}]_n &:= (\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u}_1))\mathbf{n} - (\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2))\mathbf{n} = 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{2}$$

where $\mathbf{n}$ is the outer normal to Γ from Ω_1.