A Comparative Study of Stochastic Collocation Methods for Flow in Spatially Correlated Random Fields

Haibin Chang[*,1] and Dongxiao Zhang[1,2]

[1] *Department of Energy and Resources Engineering, College of Engineering, Peking University, Beijing 100871, China.*
[2] *Sonny Astani Department of Civil and Environmental Engineering, University of Southern California, Los Angeles, CA 90089, USA.*

Abstract. Stochastic collocation methods as a promising approach for solving stochastic partial differential equations have been developed rapidly in recent years. Similar to Monte Carlo methods, the stochastic collocation methods are non-intrusive in that they can be implemented via repetitive execution of an existing deterministic solver without modifying it. The choice of collocation points leads to a variety of stochastic collocation methods including tensor product method, Smolyak method, Stroud 2 or 3 cubature method, and adaptive Stroud method. Another type of collocation method, the probabilistic collocation method (PCM), has also been proposed and applied to flow in porous media. In this paper, we discuss these methods in terms of their accuracy, efficiency, and applicable range for flow in spatially correlated random fields. These methods are compared in details under different conditions of spatial variability and correlation length. This study reveals that the Smolyak method and the PCM outperform other stochastic collocation methods in terms of accuracy and efficiency. The random dimensionality in approximating input random fields plays a crucial role in the performance of the stochastic collocation methods. Our numerical experiments indicate that the required random dimensionality increases slightly with the decrease of correlation scale and moderately from one to multiple physical dimensions.

1 Introduction

Geological formation properties are ordinarily observed at a few locations despite they exhibit a high degree of heterogeneity. This leads to uncertainty in the description of the

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[*]Corresponding author. *Email addresses:* `haibinch@usc.edu` (H. Chang), `donzhang@usc.edu` (D. Zhang)

formation properties and thus in the prediction of subsurface flow and transport. Such uncertainty necessitates a stochastic description of the formation parameters, which leads to stochastic partial differential equations governing flow and transport [25].

Monte Carlo (MC) simulation is one of the most natural approaches for solving the stochastic differential equations numerically. It is a statistical method that samples a large number of realizations for the random process and approximates the moments of interest with ensemble average. Thus the number of realizations, which one chooses, controls the accuracy of MC simulation [4]. To ensure the convergence of the moments, typically a few thousand samples or more are required, which is the main disadvantage of the direct sampling MC simulation.

An alternative approach is the stochastic finite element method, which has been rapidly developed in recent years [2, 6, 9, 10]. This method employs the polynomial chaos expansion (PCE) for random processes. After truncation in probability space, its formulation fits into the traditional spectral method framework [5, 9]. However, as the deterministic spectral methods, one must solve a set of coupled equations for the deterministic coefficients of the PCE. This increases the computational effort when the number of coefficients is large.

To overcome the difficulty for solving the coupled system, Mathelin et al. [13] proposed the so-called stochastic collocation method (SCM), which has had several successful applications [1, 15, 22]. In this approach, the output random field is approximated by Lagrange polynomial interpolation in probability space. One can derive an uncoupled system to solve the function values at selected positions. The solution process is highly parallelizable and it is found to be quite promising approach on the basis of examples with low random dimensions in the input random fields. The choice of collocation points leads to a variety of collocation methods including tensor product method [1], Smolyak method [15], Stroud 2 or 3 cubature method [17], and adaptive Stroud method [7].

Another kind of collocation method is the probabilistic collocation method (PCM) introduced by Tatang et al. [18] and successfully applied to the uncertainty analysis in some fields [11, 12, 20]. In this approach, the polynomial chaos expansion is used to approximate the output random field in probability space. The PCM is used to determine the coefficients of the polynomial chaos expansion by solving for the output random field for different sets of collocation points. The solution process is also highly parallelizable. For the cases examined with low random dimensions, this approach is found to be accurate and computationally efficient.

For numerical methods, accuracy and efficiency are two important aspects. For both the SCM and PCM, the computational efficiency depends on the total number of collocation points, which depends on both the representative random dimensions in the input random fields and the order of polynomial or other expansions in the dependent random fields. For a given dimensionality of input random field, the effect of the order or level of approximations in representing the dependent random fields has been studied [1,12,22]]. However, in practical applications the exact or proper dimension of the random space is often not known *a priori*. When the underlying (input) fields are spatially correlated, they