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Application of the Local Discontinuous Galerkin Method for the Allen-Cahn/Cahn-Hilliard System

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Abstract. In this paper, we consider the application of the local discontinuous Galerkin method for the Allen-Cahn/Cahn-Hilliard system. The method in this paper extends the local discontinuous Galerkin method in [10] to the more general application system which is coupled with the Allen-Cahn and Cahn-Hilliard equations. Similar energy stability result as that in [10] is presented. Numerical results for the nonlinear problems which include the Allen-Cahn/Cahn-Hilliard system for one-dimensional and two-dimensional cases demonstrate the accuracy and capability of the numerical method.

AMS subject classifications: 65M60, 35K55

Key words: Allen-Cahn/Cahn-Hilliard system, local discontinuous Galerkin method, free energy, stability.

1 Introduction

In this paper, we consider the extension of the local discontinuous Galerkin (LDG) method in [10] for the more general Allen-Cahn/Cahn-Hilliard (AC/CH) system in $\Omega\in\mathbb{R}^d$ $(d\leq 3)$

$$\begin{cases} u_t=\nabla\cdot[b(u,v)\nabla(\Psi_u(u,v)-\gamma\Delta u)], \\ \rho v_t=-b(u,v)[\Psi_v(u,v)-\gamma\Delta v], \end{cases} \tag{1.1}$$

where γ, ρ are given constants. The mobility, $b(u,v)$, is assumed to be nonnegative and to vanish at the "pure phases" (i.e., $u=0$ or $u=1$). This assumption, which reflects divergence of the time scale in a minimal entropy completely ordered system, implies

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degeneracy of the parabolic system (1.1). The homogeneous free energy, Ψ, will be assumed to contain two terms, one which reflects entropy contribution and another which accounts for the energy of mixing.

In [10], we developed the LDG methods for the Cahn-Hilliard equations which use the discontinuous, piecewise polynomial as the solution and test functions. In this paper, we extend the techniques in [10] to devise an LDG method for the AC/CH system (1.1) and state similar energy stability result as that in [10]. The Cahn-Hilliard equations considered in [10] is a special case of the AC/CH system (1.1) when we set the initial condition with $v=0$. Numerical simulation results demonstrate that the LDG method is a very powerful method for solving this type of fully nonlinear problems.

A systematic derivation of the system (2.1) has been given by Cahn and Novick-Cohen [3], based on energetic exchange probabilities for a Fe-Al binary alloy system on a large but finite BCC lattice. The conserved and non-conserved order parameters u and v may be defined as

$$u(\boldsymbol{n})=\frac{1}{2N}\sum_{\boldsymbol{a}\in \boldsymbol{A}} c(\boldsymbol{n}+\boldsymbol{a})+c(\boldsymbol{n}), \quad v(\boldsymbol{n})=\frac{1}{2N}\sum_{\boldsymbol{a}\in \boldsymbol{A}} c(\boldsymbol{n}+\boldsymbol{a})-c(\boldsymbol{n}),$$

where $c(\boldsymbol{n})$ represents the probability of finding an Fe atom at site $\boldsymbol{n}$ of a given lattice segments and $\boldsymbol{A}$ represents the set of nearest neighbors with $\boldsymbol{N}=|\boldsymbol{A}|$. Thus the u and v of the system (2.1) satisfy the constrains

$$u\in[0,1], \quad v\in[-\frac{1}{2},\frac{1}{2}], \quad (u\pm v)\in[0,1]. \tag{1.2}$$

In [1,2], mixed finite element methods have been developed to approximate the AC/CH system and C^0 basis functions are used.

The discontinuous Galerkin (DG) method we discuss in this paper is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables. DG methods are well suited for parallel computing and *hp*-adaptation, which consists of local mesh refinement and/or the adjustment of the polynomial order in individual elements. More general information about DG methods can be found in [4,6–8].

The main motivation for the algorithm discussed in [10] and generalized in this paper originates from the LDG techniques which have been developed for convection diffusion equations (containing second derivatives) [5] and nonlinear wave equations with high order derivatives (e.g. in [9,11–13]). In these papers, stable LDG methods for quite general nonlinear wave equations including multi-dimensional and system cases have been developed. The proof of the nonlinear L^2 stability of these methods are usually given and successful numerical experiments demonstrate their capability. These results indicate that the LDG method is a good tool for solving nonlinear equations in mathematical physics.

The outline of this paper is as follows. In Section 2, we review the properties of the AC/CH system and important application areas for this system. In Section 3, we present