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A Spectral Element/Laguerre Coupled Method to the Elliptic Helmholtz Problem on the Half Line†

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Abstract. A Legendre spectral element/Laguerre coupled method is proposed to numerically solve the elliptic Helmholtz problem on the half line. Rigorous analysis is carried out to establish the convergence of the method. Several numerical examples are provided to confirm the theoretical results. The advantage of this method is demonstrated by a numerical comparison with the pure Laguerre method.

Key words: Spectral method; Helmholtz problem; unbounded domain.

AMS subject classifications: 65M70, 65N12, 65N35, 65N25

1 Introduction

Spectral methods are essentially discretization methods for the approximate solution of partial-differential equations expressed in a weak form. The most attractive property of spectral methods may be that when the solution of the problem is infinitely smooth, the convergence of the spectral method is exponential. Due to this advantage, spectral methods have achieved great success and become popular. Most of these methods have been concentrated on problems in bounded domains. There is however a need to consider approximations to PDEs in unbounded domains. In fact, many problems in science and engineering are set in unbounded domains. Some methods for such problems using Laguerre polynomials have been developed by several authors, such as Mavriplis [13], Coulaud et al. [7], Iranzo and Falqués [11], Shen [15] and Guo and Shen [9]. Generally, the larger is the degree of the Laguerre polynomial used to approximate the solutions, the smaller is the error of numerical solutions. However, there are some difficulties stemming essentially from the fact that Laguerre polynomials have very poor resolution properties as compared with other types of orthogonal polynomials (see Gottlieb and Orszag [8]). Firstly, from a theoretical point of view, due to the appearance of the weight function e^{-x}, the errors of the Laguerre spectral method are measured in the weighted spaces. Thus for large x, the

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real errors might be big even the weighted errors are small. Secondly, from a practical point of view, the distance between the adjacent Laguerre Gauss-Radau points increases very fast as the polynomial degree increases, thus the numerical solution may fit badly the exact solution near the endpoint of the node distribution. As a result, using Laguerre polynomials in whole domain is not efficient for practical computations.

To overcome this inefficiency, Wang and Guo [17] proposed a stair Laguerre method for a model equation (elliptic Helmholtz equation). Their method consists in using progressively Laguerre method within a series of subintervals. However this method seems to be unable to completely overcome the difficulty due to the non-equilibrium of the Laguerre node distribution. Guo and Ma [10] proposed a composite Laguerre-Legendre method for approximating problems in unbounded domains. But this method suffers still from the inefficiency if we want to compute an accurate solution with presence of high frequency in large domain.

In this paper, we try to generalize the idea of Guo and Ma [10] by proposing a Legendre spectral element/Laguerre coupled method for the elliptic Helmholtz problem. Our method combines the good properties of the spectral element method with the advantage of the Laguerre method in dealing with unbounded domains. Furthermore our coupled method will especially be suitable for problems possessing high frequency solutions.

The outline of this paper is as follows: in Section 2 we present the model problem and its formulation. In Section 3 we give a detailed description of our spectral element/Laguerre coupled method. The error estimations for some basic projection operators are also established. The convergence analysis is carried out in Section 4, where an error estimate for the numerical solution is provided. In Section 5 a new implementation strategy using the nodal basis is introduced. In Section 6, we present some numerical results to confirm the theoretical analysis. Accuracy comparison with the pure Laguerre method is included.

2 Problem and formulation

Let $\Lambda = (0, \infty)$, $\chi(x)$ is certain weight function in the usual sense. For any $1 \le p \le \infty$, define

$$L^p_\chi(\Lambda) = \{v;\ \|v\|_{L^p_{\chi,\Lambda}} < \infty\}$$

where

$$\|v\|_{L^p_{\chi,\Lambda}} = \begin{cases} \left(\displaystyle\int_\Lambda |v|^p \chi(x)\mathrm{d}x\right)^{\frac{1}{p}}, & \text{if} \quad 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & \text{if} \quad p = \infty. \end{cases}$$

In particular, we denote by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$ respectively the inner product and the norm of the space $L^2_\chi(\Lambda)$. For any nonnegative integer m, we define

$$H^m_\chi(\Lambda) = \{v;\ \partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m\}$$

equipped with the following semi-norm and norm

$$|v|_{m,\chi,\Lambda} = \|\partial^m_x v\|_{\chi,\Lambda}, \qquad \|v\|_{m,\chi,\Lambda} = \left(\sum_{k=0}^m |v|^2_{k,\chi,\Lambda}\right)^{\frac{1}{2}},$$

where $\partial^k_x v = \frac{\partial^k v}{\partial x^k}$.