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An Inf-Sup Stabilized Finite Element Method by Multiscale Functions for the Stokes Equations

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Abstract. In the paper, an inf-sup stabilized finite element method by multiscale functions for the Stokes equations is discussed. The key idea is to use a Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions. The inf-sup condition for P_1-P_0 triangular element (or Q_1-P_0 quadrilateral element) is established. The optimal error estimates of the stabilized finite element method for the Stokes equations are obtained.

AMS subject classifications: 76D05, 65N30, 35K60

Key words: stabilized finite element method; multiscale functions; Petrov-Galerkin approach; inf-sup condition.

1 Introduction

In fluid dynamics, the Stokes equations model the slow flows of incompressible fluids or alternatively isotropic incompressible elastic materials. The Stokes equations have also become an important model for designing and analyzing finite element algorithms because some of the problems encountered for solving the Navier-Stokes equations already appear in the Stokes equations which are of simpler form. In particular, it gives the right setting for studying the stability problems connected with the choice of finite element spaces for the velocity and the pressure. It is well known that the finite element spaces cannot be chosen independently when the discretization

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is based on the Galerkin variational form, because it is very important to ensure the compatibility of the approximations of velocity and pressure (see, e.g., [19]).

It is well known that the simplest conforming low order elements like the $P_1(Q_1)-P_0$ (linear(bilinear) velocity, constant pressure) element is not stable. To overcome the limitation, many kinds of stabilized finite element methods have been proposed for the Stokes or Navier-Stokes equations. Brezzi and Pitkäranta in [3] firstly proposed the stabilized finite element method for P_1-P_1 triangular element. Later, many stabilized methods have been prooposed by relaxing the incompressibility constraint (i.e., modifying the second equation of (2.3)), see, e.g., [1–3, 6, 14–16, 22]. Furthermore, a general locally stabilized mixed finite element method was provided by Kechkar and Silvester in [20]. In [12], a new locally stabilized method based on the idea of [20] containing the jump terms across the inter-element boundaries of the macro elements was derived, which is called bubble condensation procedure. A particular kind of bubble functions of the velocity space is obtained by the residual free bubble method (RFBM) (see, e.g., [1, 9]), in which the bubble functions are the solutions of a problem containing the residual of the continuous equation at the element level. At the same time, the stabilized finite element method by multiscale functions was derived by [11], and a priori error analysis can be found in [10]. A main characteristic of the above methods is to use the Petrov-Galerkin approach to split the solution into two parts, i.e., the trial function space is enriched with the bubble functions which are the solutions to a local problem containing the residual of the momentum equation and special boundary conditions so that the local problem can be solved analytically.

In the paper, we use the Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions to propose a new stabilized finite element method for the Stokes equations. Although the main idea is derived from [10] and [11], our method is different from the one in [11] because the multiscale functions are new and the jump term which is introduced to the Galerkin variational formulation can be calculated no longer on the element boundary.

The remaining part of this paper is organized as follows. In the next section, we present the general framework and derive a stabilized finite element method for the Stokes equations. We then analyze the inf-sup stable condition for $P_1(Q_1)-P_0$ element and obtain the optimal error estimate.

2 Stabilized FEM by multiscale functions

Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d{=}2$ or $3)$ with Lipschitz boundary $\partial\Omega$. We consider the following Stokes equations:

$$-\nu\Delta u + \nabla p = f, \qquad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot u = 0, \qquad \text{in } \Omega, \tag{2.2}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{2.3}$$