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RICHARDSON EXTRAPOLATION AND DEFECT CORRECTION OF FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS *

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Abstract

Asymptotic error expansions in H^1-norm for the bilinear finite element approximation to a class of optimal control problems are derived for rectangular meshes. With the rectangular meshes, the Richardson extrapolation of two different schemes and an interpolation defect correction can be applied. The higher order numerical approximations are used to generate a posteriori error estimators for the finite element approximation.

Mathematics subject classification: 65R20, 65M12, 65M60, 65N30, 76S05, 49J20.
Key words: Optimal control problem, Finite element methods, Asymptotic error expansions, Defect correction, A posteriori error estimates.

1. Introduction

The aim of this paper is to discuss the asymptotic behavior of the finite element approximation for a model optimal control problem described as follows:

$$\begin{cases} \min\limits_{u\in K}\left\{\dfrac{1}{2}||y-z_d||_H^2+\dfrac{1}{2}||u||_U^2\right\} \\ \qquad -\mathrm{div}\,(A\nabla y)=f+Bu \quad \text{in } \Omega, \\ \qquad y|_{\partial\Omega}=0, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in R^n with Lipschitz boundary $\partial\Omega$, $L^2(\Omega)$ stands for the usual L^2-inner product space, K is a nonempty closed convex set in $L^2(\Omega)$, f, $z_d \in L^2(\Omega)$, B is a continuous linear operator from $U = L^2(\Omega)$ to $L^2(\Omega)$, $H = L^2(\Omega)$, and

$$A(\cdot)=\left(a_{i,j}(\cdot)\right)_{n\times n}\in\left(L^\infty(\Omega)\right)^{n\times n},$$

such that there is a constant $\sigma > 0$ satisfying that for any vector $X = (x_1, x_2, \ldots, x_n) \in R^n$

$$X^TA(x)X\ge\sigma||X||_{R^n}^2 \quad \text{for almost all } x\in\Omega,$$

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where

$$||X||_{R^n} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

In this paper, we only consider the two dimensional problem, i.e., $n = 2$.

Problem (1.1) is crucial in many engineering applications see, e.g., [26,32]. Finite element method is one of the efficient numerical methods for solving (1.1); the literature in this aspect is huge (see, e.g., [1–3,13]). Systematic introduction to the finite element method for partial differential equations and optimal control problems are available in, for example, [10,26,32]. At present there are extensive theoretical studies of the finite element approximation for various optimal control problems, see, e.g., [1,8,35] for a priori error estimates, and [2,3,9,28,29] for a posteriori error estimates. Very recently, superconvergence has been considered in [8,11,27,31] for Galerkin finite element methods and in [8] for mixed finite element methods.

In the present paper we study two numerical approaches of higher accuracy, namely, [11,27, 31] the Richardson extrapolation schemes and an interpolation defect correction method in the H^1-norm.

As an efficient numerical method to increase the accuracy of approximations, the Richardson extrapolation has been demonstrated in [30] for the difference method, in [5–7,12,14,15,17–22,24,25,33,34,37–39] for the (Galerkin and Petrov-Galerkin) finite element method and the mixed finite element method, in [16,36] for the collocation method and the boundary element method, respectively.

The defect correction of (Galerkin and Petrov-Galerkin) finite elements by means of an interpolation postprocessing technique is another numerical method to obtain approximations of higher accuracy, which has been studied for a wide variety of models. See, for example, [4,6,17,18,21,23] and the references cited therein.

This paper is organized as follows. In Section 2, the approximation subspace and the variational formula of (1.1) are provided. Also, the asymptotic expansion of the finite element approximation is presented in this section for the future need. To the best of our knowledge, the asymptotic expansions are new in that they are obtained under the condition that the mesh is uniform in x- or y-direction (not both x- and y-direction), which is different from those presented in the previous literatures (see, e.g., [7]). Section 3 is devoted to investigating the asymptotic expansions of the exact solution to the model problem in the H^1-norm. Two numerical approaches of the Richardson extrapolation schemes are presented in Section 3. Section 4 deals with an interpolation defect correction approximation in the H^1-norm based on the results given in Section 3. Furthermore, at the ends of Sections 3 and 4, a posteriori error estimators are furnished as by-products of these numerical solutions with higher convergence rates. Some related problems are addressed in Section 5.

2. The Asymptotic Expansion

In this section we first give the weak variational formula and the finite element method for the convex distributed optimal control problem (1.1). To this end, we denote the standard Sobolev spaces by $W^{m,q}(\Omega)$ on the domain Ω with the norm $||\cdot||_{m,q}$ and the seminorm $|\cdot|_{m,q}$. Also, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $||\cdot||_m$ and the seminorm $|\cdot|_m$. We set $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. In addition, throughout the paper, C stands for a generic positive constant, independent of the mesh size h, whose specific value depends on the context