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HERMITE SCATTERED DATA FITTING BY THE PENALIZED LEAST SQUARES METHOD*

Tianhe Zhou
Department of Mathematics, Zhejiang SCI-TECH University, 310018 Hangzhou, China
Email: bartonzhouth@gmail.com
Danfu Han
Department of Mathematics, Zhejiang University, 310029 Hangzhou, China
Email: mhdf2@zju.edu.cn

Abstract

Given a set of scattered data with derivative values. If the data is noisy or there is an extremely large number of data, we use an extension of the penalized least squares method of von Golitschek and Schumaker [Serdica, **18** (2002), pp.1001-1020] to fit the data. We show that the extension of the penalized least squares method produces a unique spline to fit the data. Also we give the error bound for the extension method. Some numerical examples are presented to demonstrate the effectiveness of the proposed method.

Mathematics subject classification: 41A15, 65M60, 65N30.
Key words: Bivariate splines, Scattered data fitting, Extension of penalized least squares method.

1. Introduction

Suppose $V = \{v_i = (x_i, y_i)\}_{i=1}^N$ is a set of points lying in a domain $\Omega \subset \mathbf{R}^2$. Let $\{f_i^{\nu,\mu}, 0 \le \nu + \mu \le r, i = 1, \cdots, N\}$ be given real values. If the data is noisy or there is an extremely large number of data, it may not be appropriate to interpolate the data. This problem arises in many applications, including, e.g., surface design on airplane or car and meteorology which we will explain in our numerical examples. We will construct a function $s \in C^{r+2}(\Omega)$ which minimizes

$$P_\lambda(s) := \sum_{i=1}^N \sum_{0\le\alpha+\beta\le r} |D_x^\alpha D_y^\beta s(v_i) - f_i^{\alpha,\beta}|^2 + \lambda E_r(s),$$

where $\lambda > 0$ is a constant and $E_r(s)$ is the energy functional defined by

$$E_r(s) = \int_\Omega \left[\sum_{k=0}^{r+2} \binom{r+2}{k} \left[D_x^k D_y^{r+2-k} s \right]^2 \right] dxdy. \tag{1.1}$$

We call this the extension of the penalized least squares method. If $W_\infty^{r+2}(\Omega)$ is the standard Sobolev space and $f_i^{\nu,\mu} = D_x^\nu D_y^\mu f(x_i, y_i) + \epsilon_i^{\nu,\mu}$ for $f \in W_\infty^{r+2}$ with noisy term $\epsilon_i^{\nu,\mu}$, we derive the error bounds for the method

$$\|f - s\|_{\infty,\Omega} \le C_1 |\triangle|^{r+2} |f|_{r+2,\infty,\Omega} + C_2 \lambda \|f\|_{\infty,\Omega},$$

where $|f|_{r+2,\infty,\Omega}$ denotes the maximum norm of the $(r + 2)$nd derivative of f over Ω and $\|f\|_{\infty,\Omega}$ is the standard infinite norm. Here $|\triangle|$ is the size of the triangulation $\triangle$ which will be

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defined latter. For $r = 0$, this approach reduces to a typical penalized least squares problem (see, e.g., [5]). In [5], the error bound of the penalized least squares method is provided. We will generalize that result to Hermite data setting. For $r \geq 1$, the problem has received less attention. It is easy to see that when $\lambda \gg 1$, the surface is close to the energy minimization method and when $\lambda \ll 1$, the surface is close to the discrete least squares fitting. Consequently, we can choose an appropriate weight λ for our need (see, e.g., [12]).

The paper is organized as follows. In Sect. 2 we review some well-known Bernstein-Bézier notation. The extension of the penalized least squares method is explained in Sect. 3 and the existence and uniqueness are discussed there. In Sect. 4 we derive error bounds for the extension of the penalized least squares method. Finally, in last section numerical examples are presented to demonstrate the usefulness of our method.

2. Preliminaries

Given a triangulation $\triangle$ and integers $0 \leq m < d$, we write

$$S_d^m(\triangle) := \left\{ s \in C^m(\Omega) : s|_T \in P_d, \text{for all } \; T \in \triangle \right\}$$

for the usual space of splines of degree d and smoothness m, where P_d is the $\binom{d+2}{2}$ dimensional space of bivariate polynomials of degree d. Throughout the paper we shall make extensive use of the well-known Bernstein-Bézier representation of splines. For each triangle $T = \langle v_1, v_2, v_3 \rangle$ in $\triangle$ with vertices v_1, v_2, v_3, the corresponding polynomial piece $s|_T$ is written in the form

$$s|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

where B_{ijk}^d are the Bernstein-Bézier polynomials of degree d associated with T. In particular, if $(\lambda_1, \lambda_2, \lambda_3)$ is the barycentric coordinates of any point $u \in \mathbf{R}^2$ in terms of the triangle T, then

$$B_{ijk}^d(u) := \frac{d!}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k, \quad i + j + k = d.$$

As usual, we associate the Bernstein-Bézier coefficients $\{c_{ijk}^T\}_{i+j+k=d}$ with the domain points $\{\xi_{ijk}^T := (iv_1 + jv_2 + kv_3)/d\}_{i+j+k=d}$ and use c_ξ to denote the association.

Definition 1. *Let $\beta < \infty$. A triangulation $\triangle$ is said to be* β-quasi-uniform *provided that*

$$|\triangle| \leq \beta \rho_\triangle,$$

where $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$, and $\rho_\triangle$ is the minimum of the radii of the incircles of triangles of $\triangle$.

It is easy to see that if $\triangle$ is β-quasi-uniform, then the smallest angle in $\triangle$ is bounded below by $2/\beta$.

A determining set for a spline space $S \subseteq S_d^0(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points such that if $s \in S$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\xi = 0$ for all domain points, i.e., $s \equiv 0$. The set $\mathcal{M}$ is called a minimal determining set (MDS) for S if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for S if and only if every spline $s \in S$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$.