Journal of Computational Mathematics, Vol.26, No.2, 2008, 250–258.

THE SENSITIVITY OF THE EXPONENTIAL OF AN ESSENTIALLY NONNEGATIVE MATRIX*

Weifang Zhu, Jungong Xue and Weiguo Gao
School of Mathematical Sciences, Fudan University, Shanghai 200433, China
Email: weifan_zhu@126.com, xuej@fudan.edu.cn, wggao@fudan.edu.cn

Abstract

This paper performs perturbation analysis for the exponential of an essentially nonnegative matrix which is perturbed in the way that each entry has a small relative perturbation. For a general essentially nonnegative matrix, we obtain an upper bound for the relative error in 2-norm, which is sharper than the existing perturbation results. For a triangular essentially nonnegative matrix, we obtain an upper bound for the relative error in entrywise sense. This bound indicates that, if the spectral radius of an essentially nonnegative matrix is not large, then small entrywise relative perturbations cause small relative error in each entry of its exponential. Finally, we apply our perturbation results to the sensitivity analysis of RC networks and complementary distribution functions of phase-type distributions.

Mathematics subject classification: 65F35, 15A42.
Key words: Essentially nonnegative matrix, Matrix exponential, Entrywise perturbation theory, RC network, Phase-type distribution.

1. Introduction

The matrix exponential is an important matrix function and receives extensive attention in the literatures. Matrix exponential e^{At}, which is defined as

$$e^{At} = \sum_{k=0}^{\infty} (At)^k / k!,$$

is the unique solution to the initial value problem

$$\frac{d}{dt} X(t) = AX(t), \qquad X(0) = I.$$

Many methods have been developed to compute matrix exponentials, see [7, 9, 15, 16, 20, 22, 23, 26] and references therein. Also, much perturbation analysis has been performed, see [10, 12, 13, 24] and references therein, to assess the algorithms and estimate error bounds. However, the existing error bounds are obtained for general matrices under general perturbations, with no regard to the structures of the matrices and their perturbations. In this paper, we consider the entrywise perturbation theory for essentially nonnegative matrices.

A matrix $A = (a_{ij})_{i,j=1}^n$ is said to be an essentially nonnegative matrix if its off-diagonal entries are non-negative, i.e., $a_{ij} \geq 0$ for $i \neq j$, see [25]. The exponentials of essentially nonnegative matrices frequently arise in many research areas, such as Markov chains, queuing systems and RC networks. In this paper, for a general essentially nonnegative matrix A, we

* Received September 11, 2006 / Revised version received June 1, 2007 / Accepted September 19, 2007 /

consider how the exponential e^{At} for $t \geq 0$ is perturbed when each entry of A gets a small relative perturbation. More specifically, let $E = (e_{ij})_{i,j=1}^{n}$ be a small perturbation to A in entrywise sense, i.e., for all i and j, and some $\epsilon > 0$,

$$|e_{ij}| \leq \epsilon |a_{ij}|, \tag{1.1}$$

we present an upper bound for $\phi(t)$, which is defined as

$$\phi(t) = \frac{\|e^{(A+E)t} - e^{At}\|}{\|e^{At}\|}. \tag{1.2}$$

Here, $\|\cdot\|$ denotes 2-norm. Compared to the upper bounds obtained by applying the perturbation results of Van Loan [24] directly to the problem under consideration, our bound is tighter.

In some applications, the entries of A and its exponential e^{At} have some physical meaning. It is of great interest to study how the individual entries of e^{At} are perturbed when each entry of A gets a small relative perturbation. The exponentials of triangular essentially nonnegative matrices play an important role in representation of phase-type distributions. In this paper, for a triangular essentially nonnegative matrix A, we present an upper bound for $\psi(t)$, which is defined as

$$\psi(t) = \max_{(e^{At})_{ij} \neq 0} \frac{|(e^{(A+E)t})_{ij} - (e^{At})_{ij}|}{|(e^{At})_{ij}|}. \tag{1.3}$$

The error bound indicates that if $\rho(A)t$, where $\rho(A)$ is the spectral radius of A, is not large, then small entrywise relative perturbation in A only causes small relative error in each entry of e^{At}.

Finally we apply our perturbation results to the sensitivity analysis of RC networks and complementary distribution functions of phase-type distributions with triangular representation.

Throughout this paper, $\|\cdot\|$ denotes 2-norm. For a matrix $X = (x_{ij})$, we denote by $|X|$ the matrix of entries $|x_{ij}|$ and by $X \geq Y$, where $Y = (y_{ij})$ is of identical dimension as X, if $x_{ij} \geq y_{ij}$ for all i and j. Especially, $X \geq 0$ means that every entry of X is nonnegative. These symbols are also applicable to row and column vectors. In accordance with this convention, Eq. (1.1) can be written as

$$|E| \leq \epsilon |A|. \tag{1.4}$$

2. Perturbation Bound in 2-norm

In this section, we will obtain an upper bound for $\phi(t)$ in (1.2). To this end, we need the following identity which appeared in [1]:

$$e^{(A+E)t} = e^{At} + \int_0^t e^{A(t-s)} E e^{(A+E)s} ds. \tag{2.1}$$

We first obtain the upper bound for the case that A is nonnegative.

Lemma 2.1. *Let A be an $n \times n$ nonnegative matrix and E a perturbation matrix to A satisfying $|E| \leq \epsilon A$. Then*

$$\phi(t) \leq \epsilon \|A\| t e^{\epsilon \|A\| t}.$$